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Some systematics of the coupling constant dependence of $N = 4$ Yang-Mills

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Abstract: The operator, $\mathcal{O}_\tau$, that generates infinitesimal changes of the coupling constant in $N = 4$ Yang-Mills sits in the same supermultiplet as the superconformal currents. We show how superconformal current Ward identities determine a class of terms in the operator product expansion of $\mathcal{O}_\tau$ with any other operator. In certain cases, this leads to constraints on the coupling dependence of correlation functions in $N = 4$ Yang-Mills. As an application, we demonstrate the exact non-renormalization of two and certain three-point correlation functions of BPS operators.

Keywords: Space-Time Symmetries, Extended Supersymmetry.

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1. Introduction

Maximally supersymmetric Yang-Mills in 4 dimensions has played a central role in the development of duality in both field theory and string theory. At loci in the Coulomb branch where some non-abelian gauge symmetry is unbroken, there exists an interacting superconformal field theory parametrized by a coupling constant, τ, given in terms of the Yang-Mills coupling constant and theta-angle by

$$\tau = \frac{\theta_{\rm YM}}{2\pi} + \frac{4\pi i}{g_{\rm YM}^2}. \tag{1.1}$$

The superconformal field theory is defined by correlation functions of local operators. There is, by now, convincing evidence that $N = 4$ Yang-Mills is invariant under an $\mathrm{SL}(2,\mathbb{Z})$ strong-weak coupling duality group which acts on τ [1, 2]. For simply-laced gauged groups, this duality can be naturally understood by viewing $N = 4$ Yang-Mills as a torus reduction of the 6-dimensional $(2,0)$ chiral tensor theory [3]. The $\mathrm{SL}(2,\mathbb{Z})$ duality group then corresponds to the symmetry group of the compactification torus.

What this picture intuitively suggests is that the coupling constant dependence of correlation functions should be controllable to roughly the same degree as the space-time dependence. Both dependences originate from diffeomorphisms in 6 dimensions. The goal of this work is to explore the extent to which this statement can be made precise. Consider a correlator of local operators,

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle\,. \tag{1.2}$$

To determine the coupling dependence, we want to evaluate

$$\frac{\partial}{\partial\tau}\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle\,, \tag{1.3}$$

but this has two distinct contributions. The first comes from the explicit derivative acting on each operator, while the second corresponds to the insertion of the holomorphic part of the action [4],

$$\frac{\partial}{\partial\tau}\left\langle \prod_i \mathcal{O}_i(x_i)\right\rangle = \left\langle \frac{\partial}{\partial\tau}\left(\prod_i \mathcal{O}_i(x_i)\right)\right\rangle + \frac{i}{4\tau_2}\int d^4z \left\langle \mathcal{O}_\tau(z)\prod_i \mathcal{O}_i(x_i)\right\rangle, \tag{1.4}$$

where the full action is given by

$$S = \frac{i}{4\tau_2}\int d^4z \left\{\tau\mathcal{O}_\tau(z) - \bar{\tau}\bar{\mathcal{O}}_\tau(z)\right\}. \tag{1.5}$$

What is special about $\mathcal{O}_\tau$ is that it sits in the current multiplet of $N = 4$ Yang-Mills, together with the stress-energy tensor, the supercurrents and the SU(4) R-symmetry currents [5, 4]. This is in agreement with our higher-dimensional intuition and will play a crucial role in our analysis.

Although $\mathcal{O}_\tau$ is not a current, we will show that its OPE with any other operator is special in the same way that a current OPE is special. For example, for a scalar operator $\mathcal{O}(y)$, this OPE takes the schematic form (suppressing all indices and details)

$$\begin{aligned}\mathcal{O}_\tau(x)\mathcal{O}(y) \sim\ & \frac{1}{|x-y|^2}\{Q,[Q,\{\bar{Q},[\bar{Q},\mathcal{O}(y)]\}]\} + \\ & + \frac{1}{|x-y|}\{Q,[Q,\{Q,[\bar{Q},\{\bar{Q},[\bar{Q},\mathcal{O}(y)]\}]\}]\} + \cdots .\end{aligned}$$

This expression captures all local operators in the same supermultiplet as $\mathcal{O}(y)$. It omits operators that live in different supermultiplets. The precise form of the OPE is derived in section 3.2. The OPE coefficients are, a priori, arbitrary functions of the coupling. It is important to note that the most singular term we might have expected, $1/|x-y|^4$, does not appear on the right hand side.

The determination of this OPE leads to a number of results. In this paper, we will show that the structure of the OPE together with superconformal Ward identities leads to a non-renormalization theorem for two-point functions of 1/2, 1/4 and 1/8 BPS operators. We also show the exact non-renormalization of three-point correlators of 1/2, 1/4 and 1/8 BPS superconformal primary operators. This non-renormalization result extends to three-point functions of superconformal descendents of the current multiplet via the results of [6]. However, it is difficult to extend our argument to three-point functions of generic BPS operators. In cases where there are no instanton corrections to a given correlator, we give an argument in section 4.4.1 for exact non-renormalization. It is an interesting open question to prove (or disprove) exact non-renormalization for generic BPS three-point correlators.

This non-renormalization result, originally conjectured for 1/2 BPS superconformal primary operators in [7] and verified at 1-loop in [8, 9], has been argued using an on-shell superspace formalism [10]–[16]. The non-renormalization of two and three-point functions involving 1/4 BPS operators has been conjectured and verified at 1-loop in [17, 18].

In section 4.4.2, we show that the OPE coefficients between $\mathcal{O}_\tau$ and any BPS operator are not renormalized. This leads to a pretty formula for the integrated OPE between $\mathcal{O}_\tau$ and a BPS operator $\mathcal{O}$,

$$\int d^4z\,\mathcal{O}_\tau(z)\mathcal{O}(x) \sim \sum_i \mathcal{O}'_i(x) + \cdots , \tag{1.6}$$

where $\mathcal{O}'_i$ is also BPS and sits in the same supermultiplet as $\mathcal{O}$. The omitted terms involve long and semi-short operators. A much stranger renormalization result will be described in a sequel [19]. It might also be possible to use our results to study the bonus $\mathrm{U}(1)_Y$ predictions described in [4, 20] and perhaps study extremal correlators [21].

2. Short and current multiplets

2.1 The structure of the multiplets

The superconformal symmetry group of $N = 4$ Yang-Mills is generated by 16 real supersymmetry generators which we denote by $Q_{i\alpha}, \bar{Q}^i_{\dot{\alpha}}$ where $i = 1, \ldots, 4$ and $\alpha = 1, 2$. Our notation closely follows that of [22]. In addition, there are 16 superconformal charges $S^{i\alpha}, \bar{S}^{\dot{\alpha}}_i$ and an $\mathrm{SU}(4)_R$ symmetry with generators R^i_j. The structure of the superconformal algebra is given in appendix A.

There are superconformal multiplets of different sizes. A multiplet contains a state of lowest conformal dimension which is annihilated by both the superconformal charges and by the generators of special conformal transformations, K_μ. These properties define the (unique) superconformal primary state, which we characterize by its $\mathrm{SU}(4)_R$ Dynkin labels $[k, p, q]$ $(k, p, q \geq 0)$ and its spin quantum numbers $(j, \bar{\jmath})$ under $\mathrm{Spin}(3,1) \approx \mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R$. This state cannot be obtained by acting on any other state with combinations of $Q_{i\alpha}$ or $\bar{Q}^i_{\dot{\alpha}}$. We denote this state by $|k, p, q; j, \bar{\jmath}\rangle^{\mathrm{hw}}$.

By acting on this state with $Q_{i\alpha}, \bar{Q}^i_{\dot{\alpha}}$, we generate the remaining states in the multiplet. These states (including the superconformal primary) are conformal primaries. Acting further on these states with P_μ generates conformal descendents. In this way, we construct the entire multiplet. If the superconformal primary is annihilated by some of the supersymmetry generators, the multiplet is reduced in size. It is akin to a BPS particle. For example, if all the supercharges kill the superconformal primary then it must be the unique vacuum state. Standard short representations are annihilated by 8 supersymmetry charges (hence $\frac{1}{2}$ BPS) [23]. If the superconformal primary is not annihilated by any supercharges then the multiplet is long.

For a short representation, following [22], we choose a basis where

$$Q_{i\alpha}|k, p, q; j, \bar{\jmath}\rangle^{\mathrm{hw}} = \bar{Q}^j_{\dot{\alpha}}|k, p, q; j, \bar{\jmath}\rangle^{\mathrm{hw}} = 0\,, \tag{2.1}$$

for $i = 1, 2$ and $j = 3, 4$. We are free to act on the superconformal primary state by any of the remaining 8 supersymmetry charges. As derived in appendix B, a superconformal primary has quantum numbers $[0, p, 0]_{(0,0)}$ in the terser notation $[k, p, q]_{(j,\bar{\jmath})}$ and conformal dimension $\Delta = p$.

The $Q, \bar{Q}$ operators can be used to build a multiplet 'up' by acting on the superconformal primary. On the other hand, the $S, \bar{S}$ operators move us 'down' the multiplet. We summarize the structure of the multiplet pictorially in diagram (2.2); the details of the construction appear in appendix B. The $\swarrow$ denotes the action of Q while $\searrow$ denotes the action of $\bar{Q}$) [6],

$$
\begin{array}{ccccccccccc}
 & & & & & [0,p,0]_{(0,0)} & & & & & \\
 & & & & \swarrow & & \searrow & & & & \\
 & & & [0,p-1,1]_{(\frac{1}{2},0)} & & & & [1,p-1,0]_{(0,\frac{1}{2})} & & & \\
 & & \swarrow & & \searrow & & \swarrow & & \searrow & & \\
 & & \substack{[0,p-2,2]_{(0,0)}\\ [0,p-1,0]_{(1,0)}} & & & [1,p-2,1]_{(\frac{1}{2},\frac{1}{2})} & & & \substack{[2,p-2,0]_{(0,0)}\\ [0,p-1,0]_{(0,1)}} & & \\
 & \swarrow & \searrow & & \swarrow & & \searrow & & \swarrow & \searrow & \\
 & [0,p-2,1]_{(\frac{1}{2},0)} & & \substack{[1,p-3,2]_{(0,\frac{1}{2})}\\ [1,p-2,0]_{(1,\frac{1}{2})}} & & & & \substack{[2,p-3,1]_{(\frac{1}{2},0)}\\ [0,p-2,1]_{(\frac{1}{2},1)}} & & [1,p-2,0]_{(0,\frac{1}{2})} & \\
\swarrow & \searrow & \swarrow & \searrow & & & \swarrow & \searrow & \swarrow & & \searrow \\
[0,p-2,0]_{(0,0)} & & [1,p-3,1]_{(\frac{1}{2},\frac{1}{2})} & & \substack{[2,p-4,2]_{(0,0)}\\ [0,p-3,2]_{(0,1)}} & & \substack{[0,p-2,0]_{(1,1)}\\ [2,p-3,0]_{(1,0)}} & & [1,p-3,1]_{(\frac{1}{2},\frac{1}{2})} & & [0,p-2,0]_{(0,0)} \\
\searrow & \swarrow & \searrow & \swarrow & & & \searrow & \swarrow & \searrow & & \swarrow \\
 & [1,p-3,0]_{(0,\frac{1}{2})} & & \substack{[0,p-3,1]_{(\frac{1}{2},1)}\\ [2,p-4,1]_{(\frac{1}{2},0)}} & & & & \substack{[1,p-3,0]_{(1,\frac{1}{2})}\\ [1,p-4,2]_{(0,\frac{1}{2})}} & & [0,p-3,1]_{(\frac{1}{2},0)} & \\
 & \searrow & \swarrow & & \searrow & & \swarrow & & \searrow & \swarrow & \\
 & & \substack{[0,p-3,0]_{(0,1)}\\ [2,p-4,0]_{(0,0)}} & & & [1,p-4,1]_{(\frac{1}{2},\frac{1}{2})} & & & \substack{[0,p-3,0]_{(1,0)}\\ [0,p-4,2]_{(0,0)}} & & \\
 & & \searrow & & \swarrow & & \searrow & & \swarrow & & \\
 & & & [1,p-4,0]_{(0,\frac{1}{2})} & & & & [0,p-4,1]_{(\frac{1}{2},0)} & & & \\
 & & & & \searrow & & \swarrow & & & & \\
 & & & & & [0,p-4,0]_{(0,0)} & & & & &
\end{array}
\tag{2.2}
$$

The dimension of this short representation is equal to $\frac{64}{3}p^2(p^2-1)$. The conformal dimensions range from p to $p+4$.

Let us now discuss the particular case of $p = 2$ which is central to our later discussion. We shall also see how multiplet shortening occurs for this multiplet. This is a 256-dimensional representation consisting of 128 bosonic and 128 fermionic degrees of freedom. The multiplet is given by the following diagram [22]

$$
\begin{array}{ccccccccc}
 & & & & [0,2,0]_{(0,0)} & & & & \\
 & & & \swarrow & & \searrow & & & \\
 & & [0,1,1]_{(\frac{1}{2},0)} & & & & [1,1,0]_{(0,\frac{1}{2})} & & \\
 & \swarrow & & \searrow & & \swarrow & & \searrow & \\
 & \substack{[0,0,2]_{(0,0)}\\ [0,1,0]_{(1,0)}} & & & [1,0,1]_{(\frac{1}{2},\frac{1}{2})} & & & \substack{[2,0,0]_{(0,0)}\\ [0,1,0]_{(0,1)}} & \\
\swarrow & \searrow & & \swarrow & & \searrow & \swarrow & & \searrow \\
[0,0,1]_{(\frac{1}{2},0)} & & [1,0,0]_{(1,\frac{1}{2})} & & & & [0,0,1]_{(\frac{1}{2},1)} & & [1,0,0]_{(0,\frac{1}{2})} \\
\swarrow & & & \searrow & & \swarrow & & & \searrow \\
[0,0,0]_{(0,0)} & & & & [0,0,0]_{(1,1)} & & & & [0,0,0]_{(0,0)}
\end{array}
\tag{2.3}
$$

Note that there is a difference between diagrams (2.2) (for $p = 2$) and (2.3), which we now explain. First, there are 12 representations in (2.2) which involve $p-3$ as a Dynkin label and they are absent for $p = 2$. This actually follows from the Racah-Speiser algorithm discussed in appendix B. One can show that all SU(4) representations characterized by a highest weight state with -1 as a Dynkin label vanish using (B.13) in the tensor product decomposition (B.12).

Second, there are 9 representations with $p-4$ as a Dynkin label. For $p=2$, using the Racah-Speiser algorithm, it can be shown that 5 of them vanish in (2.2) and so do not appear in (2.3). The 4 surviving ones are: $[2,-2,2]_{(0,0)}$ with $\Delta=4$, $[2,-2,1]_{(\frac{1}{2},0)}$ and $[1,-2,2]_{(0,\frac{1}{2})}$ with $\Delta=\frac{9}{2}$ and $[1,-2,1]_{(\frac{1}{2},\frac{1}{2})}$ with $\Delta=5$ and we have removed these representations from (2.3) leading to multiplet shortening.

The reason we remove these representations is that they vanish after we impose current conservation. In constructing the on-shell short multiplet, we will impose current conservation. Without the on-shell condition, the dimension of the multiplet would be incorrect. This will become explicit when we discuss the operators corresponding to this representation. As an aside, note that for $p>2$, there is no multiplet shortening. For $p=3$, the representations in (2.2) which have $p-4$ as a Dynkin label vanish directly by the Racah-Speiser algorithm; we do not have to impose the equations of motion. For higher powers of p, all the Dynkin labels which appear in (2.2) are non-negative and again there is no multiplet shortening.

2.2 More on the current multiplet

The $p=2$ multiplet is the current multiplet [24]. We will need to identify operators with the states of (2.3). The current multiplet contains the energy momentum tensor $T_{\mu\nu}\sim[0,0,0]_{(1,1)}$, the supersymmetry currents $J^{\mu}_{i\alpha}\sim[1,0,0]_{(1,\frac{1}{2})}$ and their conjugates $\bar{J}^{\mu i}_{\dot{\alpha}}\sim[0,0,1]_{(\frac{1}{2},1)}$ and the R-symmetry currents $R^{\mu i}{}_{j}\sim[1,0,1]_{(\frac{1}{2},\frac{1}{2})}$.

Apart from the currents mentioned above, the bosonic operators in the multiplet are the real scalars $Q^{[ij]}{}_{[kl]}\sim[0,2,0]_{(0,0)}$, the complex scalars $\mathcal{E}^{(ij)}\sim[0,0,2]_{(0,0)}$, $\mathcal{O}_\tau\sim[0,0,0]_{(0,0)}$, their conjugates $\bar{\mathcal{E}}_{(ij)}\sim[2,0,0]_{(0,0)}$ and $\bar{\mathcal{O}}_\tau\sim[0,0,0]_{(0,0)}$. Lastly, there is an antisymmetric 2-form $B^{[ij]}_{\mu\nu}\sim[0,1,0]_{(1,0)}$ and its conjugate $\bar{B}_{[ij]\mu\nu}\sim[0,1,0]_{(0,1)}$. It is critical that $\mathcal{O}_\tau$ sits in this multiplet.

The remaining fermionic operators in the multiplet are the spin-$\frac{1}{2}$ fermions $\chi^{i}{}_{jk}\sim[0,1,1]_{(\frac{1}{2},0)}$, $\Lambda^i\sim[0,0,1]_{(\frac{1}{2},0)}$ and their conjugates $\bar{\chi}_i{}^{jk}\sim[1,1,0]_{(0,\frac{1}{2})}$ and $\bar{\Lambda}_i\sim[1,0,0]_{(0,\frac{1}{2})}$. These composite operators can be constructed in terms of the fundamental fields in the abelian theory [24]. However, for most of our discussion, we will not need the classical expressions for the operators (note that some of the operators for the non-abelian theory have been written down in [25]).

Current conservation leads to multiplet shortening as discussed before. To see this note that

$$\begin{aligned}\partial^\mu T_{\mu\nu}&\sim[1,-2,1]_{(\frac{1}{2},\frac{1}{2})}\,, &\qquad \partial_\mu J^{\mu}_{i\alpha}&\sim[2,-2,1]_{(\frac{1}{2},0)}\,,\\ \partial_\mu\bar{J}^{\mu i}_{\dot{\alpha}}&\sim[1,-2,2]_{(0,\frac{1}{2})}\,, &\qquad \partial_\mu R^{\mu i}{}_{j}&\sim[2,-2,2]_{(0,0)}\,.\end{aligned}$$

The conservation of these currents leads to the vanishing of these four representations.

There are certain aspects of the current multiplet which will be useful later. We restrict ourselves to the free abelian theory for this part of the discussion. This restriction imposes no loss of generality since the operators in the non-abelian theory must satisfy the same algebra. The supersymmetry transformations in this case (with gauge covariant derivatives going over to ordinary derivatives and with all commutators set to zero) are given by

$$\begin{aligned}
\hat\delta\varphi^{ij} &= \frac{1}{2}(\lambda^i\eta^j - \lambda^j\eta^i) + \frac{1}{2}\epsilon^{ijkl}\bar\eta_k\bar\lambda_l, \\
\hat\delta\lambda^i{}_\alpha &= -\frac{1}{2}(\sigma^{\mu\nu})_\alpha{}^\beta F_{\mu\nu}\eta^i{}_\beta + 4i\partial_{\alpha\dot\alpha}\varphi^{ij}\bar\eta_j{}^{\dot\alpha}, \\
\hat\delta A_\mu &= -i(\lambda^i\sigma_\mu\bar\eta_i + \eta^i\sigma_\mu\bar\lambda_i)\,,
\end{aligned} \tag{2.4}$$

where η is a Grassmann parameter. It will be important to remember that the supersymmetry transformations are independent of the coupling with our choice of action (1.5). We can construct the supersymmetry transformations of the current multiplet using (2.4), which we display [24]

$$\begin{aligned}
\hat\delta Q^{ij}{}_{kl} &= \frac{4}{3}i\bar\eta_{[k}\bar\chi_{l]}{}^{ij} - \frac{2}{3}i\delta^{[i}{}_{[k}\bar\eta_m\bar\chi_{l]}{}^{mj]} + \text{h.c.}\,, \\
\hat\delta\chi^k{}_{ij} &= \frac{3}{4}\Big\{i\epsilon_{ijmn}\sigma^{\mu\nu}B^{km}_{\mu\nu}\eta^n + i\epsilon_{ijmn}\mathcal{E}^{mk}\eta^n - \\
&\qquad - i\sigma^\mu R^k{}_{\mu[i}\bar\eta_{j]} + 2i\sigma^\mu\partial_\mu Q^{kl}{}_{ij}\bar\eta_l\Big\} - \text{trace}\,, \\
\hat\delta B^{ij}_{\mu\nu} &= -\frac{1}{2}\epsilon^{ijkl}\bar\eta_k\bar\sigma^\rho\sigma_{\mu\nu}J_{\rho l} - \eta^{[i}\sigma_{\mu\nu}\Lambda^{j]} + \frac{2}{3}i\epsilon^{ijkl}\bar\eta_n\bar\sigma^\rho\sigma_{\mu\nu}\partial_\rho\chi^n{}_{kl}\,, \\
\hat\delta\mathcal{E}^{ij} &= \eta^{(i}\Lambda^{j)} + \frac{2}{3}i\epsilon^{mnk(i}\bar\eta_k\bar\sigma^\mu\partial_\mu\chi^{j)}{}_{mn}\,, \\
\hat\delta R^{\mu i}{}_j &= -\eta^i J^\mu_j + \frac{1}{4}\delta^i{}_j\eta^k J^\mu_k + \frac{8}{3}i\eta^k\sigma^{\mu\nu}\partial_\nu\chi^i{}_{jk} - \text{h.c.}\,, \\
\hat\delta J_{\mu i} &= -\sigma^\nu T_{\mu\nu}\bar\eta_i - 2\left(\sigma_\rho\bar\sigma_{\mu\nu} - \frac{1}{3}\sigma_{\mu\nu}\sigma_\rho\right)\partial^\nu R^{\rho k}{}_i\bar\eta_k - \\
&\qquad - \left(\sigma_{\rho\sigma}\sigma_{\mu\nu} + \frac{1}{3}\sigma_{\mu\nu}\sigma_{\rho\sigma}\right)\epsilon_{iklm}\partial^\nu B^{kl\rho\sigma}\eta^m\,, \\
\hat\delta\Lambda^i &= -i\eta^i\mathcal{O}_\tau + \sigma_\mu\partial^\mu\mathcal{E}^{ik}\bar\eta_k - \sigma^{\mu\nu}\partial_\rho B^{ij}_{\mu\nu}\sigma^\rho\bar\eta_j\,, \\
\hat\delta T_{\mu\nu} &= \eta^i\sigma_{(\mu|\rho|}\partial^\rho J_{\nu)i} + \text{h.c.}\,, \\
\hat\delta\mathcal{O}_\tau &= 2i\bar\eta_i\bar\sigma^\mu\partial_\mu\Lambda^i + i\eta^i(\partial^\mu J_{\mu i} + 2\sigma^\mu\bar\sigma^\nu\partial_\mu J_{\nu i})\,,
\end{aligned} \tag{2.5}$$

There are similar expressions for the conjugate operators. Note that the terms $\eta^i\partial^\mu J_{\mu i}$ and $\eta^i\sigma^\mu\bar\sigma^\nu\partial_\mu J_{\nu i}$ in the variation of $\mathcal{O}_\tau$ do not appear in [24]. This is because both these terms vanish classically by current conservation and the spin-1/2 anomaly cancellation condition. However, both these conditions are violated in the quantum theory; this can be seen by studying Ward identities where the violation is caused by contact terms. We will analyze this violation in considerable detail in the following section where it will become clearer why we require these additional terms in the supervariation of $\mathcal{O}_\tau$.

There are a few points worth highlighting in (2.5). First, as expected, the current multiplet varies into itself. Also, note that $B^{ij}_{\mu\nu}$ transforms into J^μ_i, while $\mathcal{E}^{ij}$ does not; this will also be important later. The variation of $Q^{ij}{}_{kl}$ contains no conformal descendent. On the other hand, $T_{\mu\nu}, \mathcal{O}_\tau$ and $\bar{\mathcal{O}}_\tau$ vary only into conformal descendents. The remaining operators vary into combinations of primaries and descendents.

From the supersymmetry transformations, we see that it is easy to recover (2.3). For example, from the variation $\hat{\delta}\chi^i{}_{jk}$, we see that Q acting on $\chi^i{}_{jk}$ gives $B^{ij}_{\mu\nu}$ and $\mathcal{E}^{ij}$, while acting with $\bar{Q}$ gives $R^{\mu i}{}_j$. The other term in $\hat{\delta}\chi^i{}_{jk}$ is a conformal descendent.

2.3 The structure of $\delta\mathcal{O}_\tau$

As a matter of notation, we will denote the superconformal primary in the representation $[0,p,0]_{(0,0)}$ for arbitrary p by $\mathcal{O}_p$. We will also denote the action of Q by δ and the action of $\bar{Q}$ by $\bar{\delta}$ as in [4]. Acting on operators, $\delta^r\bar{\delta}^s$ stands for a sequence of graded commutators; for example,

$$\delta\bar{\delta}^2\mathcal{O} \leftrightarrow [Q,[\bar{Q},[\bar{Q},\mathcal{O}]_\pm]_\mp]_\pm\,.$$

Note that for all values of p, $\{Q,\bar{Q}\}$ is always zero (and never proportional to P_μ) for the particular supercharges used in diagram (2.2). Therefore, starting from $\mathcal{O}_p$, we can reach any conformal primary in the multiplet by acting suitably with δ and $\bar{\delta}$ operators in an arbitrary way (up to an overall sign), without worrying about the ordering of the operators.

For the current multiplet, it might appear from diagram (2.3) that $\delta^5 = 0$. This, however, is not the case as we will now demonstrate. Consider the two-point correlator $\langle\mathcal{O}_\tau(x)\mathcal{O}_\tau(y)\rangle$. We can express this correlator in the form

$$\langle\mathcal{O}_\tau(x)\mathcal{O}_\tau(y)\rangle = \langle\delta(\delta^3\mathcal{O}_2(x)\mathcal{O}_\tau(y))\rangle + \langle\delta^3\mathcal{O}_2(x)\delta^5\mathcal{O}_2(y)\rangle \tag{2.6}$$

where the first term on the right hand side vanishes because δ kills the vacuum. By computing both sides in the abelian theory, we can determine the structure of $\delta^5\mathcal{O}_2 = \delta\mathcal{O}_\tau$.

In the abelian theory, the operator $\mathcal{O}_\tau$ is given by

$$\mathcal{O}_\tau = \frac{\tau_2}{8\pi}\left(F_{\mu\nu}F^{\mu\nu} + iF_{\mu\nu}\widetilde{F}^{\mu\nu} - 4i\bar{\lambda}_i\bar{\sigma}_\mu\partial^\mu\lambda^i + 4\bar{\varphi}_{ij}\partial^\mu\partial_\mu\varphi^{ij}\right) \tag{2.7}$$

where $\widetilde{F} = *F$. For clarity, let us first restrict to the terms in $\mathcal{O}_\tau$ that depend only on the field strength. We will show that the supervariation of these terms gives rise to a conformal descendent (the analysis for the other terms is similar). Using the supervariations (2.4), we see that

$$\hat{\delta}\mathcal{O}_\tau = -\frac{i\tau_2}{2\pi}\left[(F_{\mu\nu} + i\widetilde{F}_{\mu\nu})(\partial^\mu\lambda^i\sigma^\nu\bar{\eta}_i + \eta^i\sigma^\nu\partial^\mu\bar{\lambda}_i)\right] + \cdots \tag{2.8}$$

where the omitted terms are generated from varying the scalars and fermions of (2.7). After some lengthy algebra, we can write this as

$$\hat{\delta}\mathcal{O}_\tau = 2i(\bar{\eta}_i\bar{\sigma}^\mu\partial_\mu\Lambda^i + \eta^i\partial^\mu J_{\mu i}) + \cdots\,, \tag{2.9}$$

which includes one of the terms of $\delta\mathcal{O}_\tau$ appearing in (2.5) and where

$$\begin{aligned}\Lambda^i{}_\alpha &= \left(\delta^3\mathcal{O}_2\right)^i{}_\alpha = -\frac{\tau_2}{4\pi}(\sigma^{\mu\nu})_\alpha{}^\beta\left(F_{\mu\nu}\lambda^i{}_\beta\right)\,,\\ J^\mu_{i\alpha} &= \left(\delta^2\bar{\delta}\mathcal{O}_2\right)^\mu_{i\alpha} = -\frac{\tau_2}{4\pi}\left(F_{\rho\sigma}(\sigma^{\rho\sigma}\sigma^\mu\bar{\lambda}_i)_\alpha + \cdots\right)\,.\end{aligned} \tag{2.10}$$

The omitted terms again involve $(\bar{\varphi},\lambda)$. From the expression for J_i in (2.10), we see that the spin-1/2 anomaly cancellation condition,

$$\bar{\sigma}_\mu J^\mu_i = 0\,, \tag{2.11}$$

is trivially satisfied without using the equations of motion and so plays no role in (2.9). We also see that

$$\partial_\mu J_i^\mu = -\frac{\tau_2}{4\pi}\partial^\mu \left[(F_{\mu\nu} + i\widetilde{F}_{\mu\nu})\sigma^\nu\bar{\lambda}_i\right] + \cdots , \tag{2.12}$$

which vanishes on-shell. After using the Bianchi identity and the equations of motion (2.12) agrees with (2.8).

Let us compute the left hand side of (2.6) in free field theory. We use the gauge field propagator in Feynman gauge given by

$$\Delta^{ab}_{\mu\nu}(x-y) = \frac{\delta^{ab}\eta_{\mu\nu}}{\pi\tau_2(x-y)^2}. \tag{2.13}$$

This satisfies the usual relation $\partial^2\Delta^{ab}_{\mu\nu}(x-y) = -\frac{4\pi}{\tau_2}\delta^4(x-y)\delta^{ab}\eta_{\mu\nu}$. Using the contraction

$$\langle F^a_{\mu\nu}(x)F^b_{\rho\sigma}(y)\rangle = \frac{\delta^{ab}}{\pi\tau_2}[\eta_{\nu\sigma}\partial^x_\mu\partial^y_\rho - \eta_{\nu\rho}\partial^x_\mu\partial^y_\sigma - \eta_{\mu\sigma}\partial^x_\nu\partial^y_\rho + \eta_{\mu\rho}\partial^x_\nu\partial^y_\sigma]\frac{1}{(x-y)^2}\,, \tag{2.14}$$

we deduce the relations

$$\begin{aligned}\langle (F^a_{\mu\nu}F^{\mu\nu a})(x)(F^b_{\rho\sigma}F^{\rho\sigma b})(y)\rangle &= \frac{16}{(\pi\tau_2)^2}\left[\left(\partial_\mu\partial_\rho\frac{1}{(x-y)^2}\right)^2 + 8\pi^4(\delta^4(x-y))^2\right],\\ \langle (F^a_{\mu\nu}\widetilde{F}^{\mu\nu a})(x)(F^b_{\rho\sigma}\widetilde{F}^{\rho\sigma b})(y)\rangle &= \frac{16}{(\pi\tau_2)^2}\left[\left(\partial_\mu\partial_\rho\frac{1}{(x-y)^2}\right)^2 - 16\pi^4(\delta^4(x-y))^2\right],\end{aligned} \tag{2.15}$$

while

$$\langle (F^a_{\mu\nu}F^{\mu\nu a})(x)(F^b_{\rho\sigma}\widetilde{F}^{\rho\sigma b})(y)\rangle = 0\,. \tag{2.16}$$

Note that up to total derivatives, the second equation in (2.15) is zero using

$$\left(\partial_\mu\partial_\rho\frac{1}{(x-y)^2}\right)^2 = \left(\partial^2\frac{1}{(x-y)^2}\right)^2 = 16\pi^4\left\{\delta^4(x-y)\right\}^2. \tag{2.17}$$

After summing these various contributions, the left hand side of (2.6) yields

$$\langle \mathcal{O}_\tau(x)\mathcal{O}_\tau(y)\rangle = \frac{3}{2}\left\{\delta^4(x-y)\right\}^2. \tag{2.18}$$

The left hand side of (2.6) is non-vanishing — in fact, it is a contact term. What about the right-hand side of (2.6)? Only the second term can be non-vanishing and indeed, $\delta^5\mathcal{O}_2$ is a conformal descendent as we see from (2.9).

From (2.8), note that

$$(\delta\mathcal{O}_\tau)_{i\alpha} = \frac{i\tau_2}{2\pi}\partial^\mu\left[(F_{\mu\nu} + i\widetilde{F}_{\mu\nu})\sigma^\nu\bar{\lambda}_i\right]_\alpha. \tag{2.19}$$

We now compute $\langle \Lambda_\alpha{}^i(x)\,(\delta\mathcal{O}_\tau)_i{}^\alpha(y)\rangle$. There is no sum over i or α, but it is easier to evaluate the correlator by summing and dividing by 8. We use the fermion propagator

$$\langle \lambda^i{}_\alpha(x)\bar{\lambda}_{j\dot\beta}(y)\rangle = \frac{i}{\pi\tau_2}(\sigma^\mu)_{\alpha\dot\beta}\partial_\mu\frac{1}{(x-y)^2}\delta^i{}_j \tag{2.20}$$

and some lengthy but straightforward algebra to obtain

$$\langle \Lambda(x)\delta\mathcal{O}_2(y)\rangle \equiv \langle \delta^3\mathcal{O}_2(x)\delta^5\mathcal{O}_2(y)\rangle = \frac{3}{2}\left\{\delta^4(x-y)\right\}^2 . \tag{2.21}$$

As had to be the case, this is a contact term in agreement with (2.18).

So from this free field analysis, we see that $\delta\mathcal{O}_\tau \sim \eta^i\partial^\mu J_{\mu i}$. It is similarly easy to find the $\eta^i\sigma^\mu\bar{\sigma}^\nu\partial_\mu J_{\nu i}$ term in $\delta\mathcal{O}_\tau$ as written in (2.5). To see this we consider the full expression for $\mathcal{O}_\tau$ in (2.7) and consider its supervariation. Including all the contributions and integrating by parts, the total contribution gives

$$\delta\mathcal{O}_\tau = i\eta^i\partial^\mu J_{\mu i} + \frac{2\tau_2}{\pi}(\eta^i\sigma^\nu\bar{\sigma}^\mu\partial_\mu\lambda^j)\partial_\nu\bar{\varphi}_{ij} . \tag{2.22}$$

We now consider the complete expression for J_i,

$$J^\mu_{i\alpha} = -\frac{\tau_2}{4\pi}\,\mathrm{Tr}\left(F_{\rho\sigma}(\sigma^{\rho\sigma}\sigma^\mu\bar{\lambda}_i)_\alpha - 2i\bar{\varphi}_{ij}\overset{\leftrightarrow}{\partial}{}^\mu\lambda^j{}_\alpha - \frac{4}{3}i\partial_\nu(\sigma^{\mu\nu}\bar{\varphi}_{ij}\lambda^j)_\alpha\right), \tag{2.23}$$

where we define $\mathcal{O}_i\overset{\leftrightarrow}{\partial}\mathcal{O}_j = \mathcal{O}_i\left(\partial\mathcal{O}_j\right) - \left(\partial\mathcal{O}_i\right)\mathcal{O}_j$. The extra terms beyond those of (2.10) do not satisfy $\bar{\sigma}^\mu J_{\mu i} = 0$ trivially; in fact, they satisfy

$$\bar{\sigma}^\mu J_{\mu i} = -\frac{i\tau_2}{\pi}\bar{\varphi}_{ij}\bar{\sigma}^\mu\partial_\mu\lambda^j , \tag{2.24}$$

which only vanishes on-shell. Using (2.24) to compute $\eta^i\sigma^\mu\bar{\sigma}^\nu\partial_\mu J_{\nu i}$, we recover the expression given in (2.22). So (2.22) becomes

$$\delta\mathcal{O}_\tau = i\eta^i\partial^\mu J_{\mu i} + 2i\eta^i\sigma^\mu\bar{\sigma}^\nu\partial_\mu J_{\nu i} , \tag{2.25}$$

giving us the desired relation in (2.5). So while $\delta\mathcal{O}_\tau$ vanishes on-shell, it is actually non-vanishing because of contact terms. These contact terms can also be seen from a Ward identity for condition (2.11) which we now deduce.

2.4 Ward identities

Let us begin by recalling the more familiar Ward identities involving the stress-energy tensor. We will use these identities in the following section. Consider an operator, Φ^I, with conformal dimension Δ_I transforming in representation I of the Lorentz group with generators $(S_{\mu\nu})^I{}_J$. The Ward identities state that,

$$\begin{aligned}
\partial^\mu\langle T_{\mu\nu}(z)\Phi^{I_1}(x_1)\cdots\Phi^{I_n}(x_n)\rangle &= \sum_{i=1}^n \partial_{\nu_i}\delta^4(z-x_i)\langle\Phi^{I_1}(x_1)\cdots\Phi^{I_n}(x_n)\rangle , \\
\langle T_{[\mu\nu]}(z)\Phi^{I_1}(x_1)\cdots\Phi^{I_n}(x_n)\rangle &= \frac{i}{2}\sum_{i=1}^n (S_{\mu\nu})^{I_i}{}_J\,\delta^4(z-x_i)\langle\Phi^{I_1}(x_1)\cdots\Phi^J(x_i)\cdots\Phi^{I_n}(x_n)\rangle , \\
\langle T^\mu_\mu(z)\Phi^{I_1}(x_1)\cdots\Phi^{I_n}(x_n)\rangle &= \sum_{i=1}^n \Delta_{I_i}\delta^4(z-x_i)\langle\Phi^{I_1}(x_1)\cdots\Phi^{I_n}(x_n)\rangle .
\end{aligned} \tag{2.26}$$

The first and the last identities express the breakdown of energy-momentum conservation and conformal invariance. The second equation shows that the stress tensor is not symmetric at the location of the operators inserted in the correlator.

The Ward identity for the supercurrent conservation, $\partial_\mu J_i^\mu = 0$, is given by

$$\partial^\mu \langle J_{\mu i}^\alpha(z)\Phi^{I_1}(x_1)\cdots\Phi^{I_n}(x_n)\rangle = \sum_{j=1}^n \delta^4(z-x_j)\left\langle \Phi^{I_1}(x_1)\cdots[Q_i{}^\alpha, \Phi^{I_j}(x_j)]_\pm\cdots\Phi^{I_n}(x_n)\right\rangle . \tag{2.27}$$

The divergence of the supercurrent is non-zero at the positions of inserted operators in the correlator.

In order to deduce the Ward identity for $\bar\sigma_\mu J_i^\mu = 0$, we first consider the Ward identity for the superconformal current $I_{\mu\alpha}^i$

$$\partial^\mu \langle I_{\mu\alpha}^i(z)\Phi^{I_1}(x_1)\cdots\Phi^{I_n}(x_n)\rangle = \sum_{j=1}^n \delta^4(z-x_j)\langle\Phi^{I_1}(x_1)\cdots[S_\alpha^i, \Phi^{I_j}(x_j)]_\pm\cdots\Phi^{I_n}(x_n)\rangle . \tag{2.28}$$

We now use the relation between these two currents given in [27] (see [28] for a recent discussion)

$$I_{\mu\alpha}^i(x) = x_{\alpha\dot\alpha}\bar J_\mu^{i\dot\alpha}(x) . \tag{2.29}$$

Inserting (2.29) into (2.28), using (2.27) and then taking the conjugate gives

$$\begin{aligned}(\bar\sigma_\mu)^{\dot\alpha\alpha}\langle J_{i\alpha}^\mu(z)\Phi^{I_1}(x_1)\cdots\Phi^{I_n}(x_n)\rangle &= \sum_{j=1}^n \delta^4(z-x_j)\langle\Phi^{I_1}(x_1)\cdots \\ &\quad\cdots[\bar S_i^{\dot\alpha} - x_j^{\dot\alpha\alpha}Q_i{}^\alpha, \Phi^{I_j}(x_j)]_\pm\cdots\Phi^{I_n}(x_n)\rangle .\end{aligned} \tag{2.30}$$

This is the desired Ward identity.

3. Constraining the $\mathcal{O}_\tau$ OPE

3.1 Overview

The strategy we will use to determine the $\mathcal{O}_\tau$ OPE makes use of the Ward identities of the superconformal currents. More precisely, we start with the stress-tensor

$$T \sim \bar\delta^2\delta^2\mathcal{O}_2 \tag{3.1}$$

whose OPE with an operator Φ is in part determined by Ward identities. What is not determined by Ward identities are local operators in the $T(z)\Phi(x)$ OPE which do not reside in the same supermultiplet as Φ. These can include long, BPS and semi-short operators. Examples of semi-short multiplets include multi-trace operators; for an example, see [26]. For example, the Konishi operator, $\mathrm{Tr}(\phi^2)$, can appear in the $T(z)T(x)$ OPE but is not determined by Ward identities. Since this operator is long, its dimension is renormalized. Therefore, whether this term is singular in the OPE depends on the value of the coupling. This same caveat will apply to all the OPEs that we determine from Ward identities.

Peeling off a $\bar{\delta}$ in (3.1) results in the OPE of $\bar{\delta}\delta^2\mathcal{O}_2$ (which is the supercurrent, $J^{\alpha}_{\mu i}$) with an arbitrary operator, modulo that caveat mentioned above. Since the OPE of $J^{\alpha}_{\mu i}$ is also determined in part by a Ward identity, this will allow us to check that the peeling off procedure works. We will then remove the remaining $\bar{\delta}$ to determine the OPE of $\delta^2\mathcal{O}_2$, which is not a current.

The next step is to construct $\delta^3\mathcal{O}_2$ by applying a δ and recursively using the known OPE results. Finally, we apply δ to obtain the OPE of $\mathcal{O}_\tau$ with any other operator. For a scalar operator, this OPE will depend on 15 coefficients $a_i(\tau,\bar{\tau})$. In section 4.4.2, we will show that these 15 coefficients are independent of the coupling for BPS operators. It should be noted that for N=1 supersymmetric theories, there exists a supermultiplet of currents [27]. The lowest component of the multiplet is the R-symmetry current, the middle component is the supercurrent, while the top component is the stress tensor. The OPE of these currents collectively can be determined in superspace. Our peeling off procedure will match these known results for the first step taking us from the stress tensor to the supercurrent. For a recent discussion of the current supermultiplet and its OPE structure, see, for example, [29].

3.1.1 The stress tensor OPE

The Ward identities (2.26) uniquely determine a class of terms in the OPE of the stress-energy tensor. The most singular terms in the OPE are given by

$$\begin{aligned} T_{\mu\nu}(z)\Phi^I(x) \sim\ & \Phi^I(x)\partial_\mu\partial_\nu\frac{1}{(z-x)^2} + \Phi^I(x)\eta_{\mu\nu}\delta^4(z-x) + \\ & +\left[\frac{i}{2}(S_{\mu\nu})^I{}_J\Phi^J(x)\delta^4(z-x) + \frac{i}{8\pi^2}\Phi^J(x)\Big(S_{\lambda\mu}\partial^\lambda\partial_\nu + S_{\lambda\nu}\partial^\lambda\partial_\mu\Big)^I{}_J\frac{1}{(z-x)^2}\right] + \\ & + \text{less singular terms}\,. \end{aligned} \tag{3.2}$$

These are all the most singular terms that involve operators in the Φ^I supermultiplet. The symbol $\sim$ in this equation indicates that we have ignored coefficients. Some of the terms appearing on the right hand side can have coupling dependent coefficients although the relative coefficient of the bracketed terms is precise. In particular, those terms that contribute to the violation of scale invariance will generally be renormalized.

Now we need to express $T_{\mu\nu}$ in terms of the supercharges and $\mathcal{O}_2$. From (2.5) we deduce the following supersymmetry transformations:

$$[Q_{p\alpha}, Q^{ij}{}_{kl}] = \frac{2}{3}i\delta^{[i}{}_{[k}\delta^m_{|p|}\chi^{j]}{}_{ml]\alpha} - \frac{4}{3}i\delta^{[i}{}_p\chi^{j]}{}_{kl\alpha}\,, \tag{3.3}$$

$$\{Q_n{}^\beta, \chi^k{}_{ij\alpha}\} = -\frac{3}{4}i\epsilon_{ijmn}\Sigma_\alpha{}^{\beta km} - \frac{3}{8}i\delta^k{}_{[i}\epsilon_{j]pmn}\Sigma_\alpha{}^{\beta pm}\,, \tag{3.4}$$

$$[\bar{Q}^{k\dot\alpha}, B^{ij}_{\mu\nu}] = -\frac{1}{2}\epsilon^{ijkl}(\bar\sigma_\rho\sigma_{\mu\nu})^{\dot\alpha\alpha}J^\rho_{l\alpha} + \frac{2}{3}i\epsilon^{ijnl}(\bar\sigma^\rho\sigma_{\mu\nu})^{\dot\alpha\alpha}\partial_\rho\chi^k{}_{nl\alpha}\,, \tag{3.5}$$

$$[\bar{Q}^{k\dot\alpha}, \mathcal{E}^{ij}] = \frac{2}{3}i\epsilon^{mnk(i}\partial_\mu\chi^{j)}{}_{mn\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\,, \tag{3.6}$$

and

$$\{\bar{Q}^j{}_{\dot\alpha}, J_{\mu i\alpha}\} = \sigma^\nu{}_{\alpha\dot\alpha}T_{\mu\nu}\delta^j{}_i + 2\left(\sigma_\rho\bar\sigma_{\mu\nu} - \frac{1}{3}\sigma_{\mu\nu}\sigma_\rho\right)_{\alpha\dot\alpha}\partial^\nu R^{\rho j}{}_i. \tag{3.7}$$

Here,

$$\Sigma_\alpha{}^{\beta ij} = \sigma_\alpha^{\mu\nu\beta} B_{\mu\nu}^{ij} + \delta_\alpha{}^\beta \mathcal{E}^{ij}\,. \tag{3.8}$$

Using these relations, after some tedious algebra, we find that

$$T_{\mu\nu} = (\sigma_\mu)_{\alpha\dot\alpha}(\sigma_\nu)_{\beta\dot\beta}\{\bar Q^{l\dot\beta},[\bar Q^{k\dot\alpha},\{Q_j{}^\beta,[Q_i{}^\alpha,Q^{ij}{}_{kl}]\}]\}. \tag{3.9}$$

We have set the irrelevent numerical factor on the right hand side of (3.9) to one. Note that the right-hand side of (3.9) is manifestly symmetric in μ and ν, using the fact that the various Q and $\bar Q$ operators anti-commute. Also (3.9) satisfies $T^\mu_\mu = 0$ by construction. These classical constraints are violated quantum mechanically in accord with (2.26).

3.1.2 The supercurrent OPE

Given the OPE (3.2), we now want to construct the supercurrent OPE by peeling off a $\bar Q$. We begin with the following relation:

$$[\bar Q^{k\dot\alpha},\{Q_j{}^\beta,[Q_i{}^\alpha,Q^{ij}{}_{kl}]\}] = \left\{(\bar\sigma^\mu)^{\dot\alpha\beta}J^\alpha_{\mu l} + (\bar\sigma^\mu)^{\dot\alpha\alpha}J^\beta_{\mu l}\right\}, \tag{3.10}$$

which leads to

$$J^\alpha_{\mu l} = (\sigma_\mu)_{\beta\dot\alpha}[\bar Q^{k\dot\alpha},\{Q_j{}^\beta,[Q_i{}^\alpha,Q^{ij}{}_{kl}]\}]\,. \tag{3.11}$$

Note that $\bar\sigma^\mu J_{\mu i} = 0$ by construction, satisfying the classical spin-1/2 anomaly cancellation condition. In deriving these relations, we do not pick up any descendents because the possible descendents involve derivatives of either $R^{\mu i}{}_i$ or $\chi^i{}_{ij\alpha}$, both of which vanish.

Now let us calculate the contribution of the three-point function to the integral

$$\int d^4z\langle T_{\mu\nu}(z)\Phi^I(x)\Phi^J(y)\rangle\,, \tag{3.12}$$

as $z\to x$ and also as $z\to y$. Considering only the most singular terms in the OPE (3.2), this reduces to

$$\begin{aligned}
\int d^4z\langle T_{\mu\nu}(z)\Phi^I(x)\Phi^J(y)\rangle \;=\; & \langle\Phi^I(x)\Phi^J(y)\rangle\left(\int_{B_x^\epsilon} d^4z\partial_\mu\partial_\nu\frac{1}{(z-x)^2} + \int_{B_y^\epsilon} d^4z\partial_\mu\partial_\nu\frac{1}{(z-y)^2}\right) + \\
& +2\langle\Phi^I(x)\Phi^J(y)\rangle\eta_{\mu\nu} + \frac{i}{2}(S_{\mu\nu})^I{}_K\langle\Phi^K(x)\Phi^J(y)\rangle + \\
& +\frac{i}{2}(S_{\mu\nu})^J{}_K\langle\Phi^I(x)\Phi^K(y)\rangle + \\
& +\frac{i}{8\pi^2}\langle\Phi^K(x)\Phi^J(y)\rangle\int_{B_x^\epsilon} d^4z\left[S_{\lambda\mu}\partial^\lambda\partial_\nu + S_{\lambda\nu}\partial^\lambda\partial_\mu\right]^I{}_K\frac{1}{(z-x)^2} + \\
& +\frac{i}{8\pi^2}\langle\Phi^I(x)\Phi^K(y)\rangle\int_{B_y^\epsilon} d^4z\left[S_{\lambda\mu}\partial^\lambda\partial_\nu + S_{\lambda\nu}\partial^\lambda\partial_\mu\right]^J{}_K\frac{1}{(z-y)^2}\,,
\end{aligned} \tag{3.13}$$

where $B_x^\epsilon(B_y^\epsilon)$ is a small ball of radius ϵ centered at $x(y)$. From now on, we shall drop the integrals for clarity and write this as

$$\langle T_{\mu\nu}(z)\Phi^I(x)\Phi^J(y)\rangle \;=\; \langle\Phi^I(x)\Phi^J(y)\rangle\partial_\mu\partial_\nu\left(\frac{1}{(z-x)^2}+\frac{1}{(z-y)^2}\right)$$

$$
\begin{aligned}
&+\eta_{\mu\nu}\langle\Phi^I(x)\Phi^J(y)\rangle\left(\delta^4(z-x)+\delta^4(z-y)\right)\\
&+(S_{\mu\nu})^I{}_K\langle\Phi^K(x)\Phi^J(y)\rangle\delta^4(z-x)+(S_{\mu\nu})^J{}_K\langle\Phi^I(x)\Phi^K(y)\rangle\delta^4(z-y)\\
&+\langle\Phi^K(x)\Phi^J(y)\rangle\left[S_{\lambda\mu}\partial^\lambda\partial_\nu+S_{\lambda\nu}\partial^\lambda\partial_\mu\right]^I{}_K\frac{1}{(z-x)^2}\\
&+\langle\Phi^I(x)\Phi^K(y)\rangle\left[S_{\lambda\mu}\partial^\lambda\partial_\nu+S_{\lambda\nu}\partial^\lambda\partial_\mu\right]^J{}_K\frac{1}{(z-y)^2}\,,
\end{aligned}
\tag{3.14}
$$

with the integrals around the various points implied. Now the left hand side of (3.14) can we rewritten using (3.9) and (3.11),

$$
\langle T_{\mu\nu}(z)\Phi^I(x)\Phi^J(y)\rangle=(\sigma_\nu)_{\beta\dot\beta}\langle J^\beta_{\mu l}(z)[\bar Q^{l\dot\beta},\Phi^I(x)\Phi^J(y)]\rangle\,. \tag{3.15}
$$

We now want to write down the OPE of $J^\alpha_{\mu i}$ with Φ^I such that the right-hand side of (3.15) yields all the terms in (3.14). We make the ansatz

$$
\begin{aligned}
J^\alpha_{\mu i}(z)\Phi^I(x)\sim{}&[Q_i{}^\alpha,\Phi^I(x)]_\pm\partial_\mu\frac{1}{(z-x)^2}+[Q_i{}^\beta,\Phi^I(x)]_\pm(\sigma_{\mu\nu})_\beta{}^\alpha\partial^\nu\frac{1}{(z-x)^2}+\\
&+(S_{\mu\nu})^I{}_J[Q_i{}^\alpha,\Phi^J(x)]_\pm\partial^\nu\frac{1}{(z-x)^2}+\\
&+(S_{\rho\lambda})^I{}_J[Q_i{}^\beta,\Phi^J(x)]_\pm(\sigma^{\rho\lambda}\sigma_{\mu\nu})_\beta{}^\alpha\partial^\nu\frac{1}{(z-x)^2}+\\
&+\Big\{(S_{\mu\nu})^I{}_J[Q_i{}^\beta,\Phi^J(x)]_\pm(\sigma^{\nu\lambda})_\beta{}^\alpha\partial_\lambda\frac{1}{(z-x)^2}+\\
&\quad+(S^{\nu\lambda})^I{}_J[Q_i{}^\beta,\Phi^J(x)]_\pm(\sigma_{\nu\mu})_\beta{}^\alpha\partial_\lambda\frac{1}{(z-x)^2}\Big\}+\\
&+\text{less singular terms}\,.
\end{aligned}
\tag{3.16}
$$

We have made this ansatz because all the terms on the right-hand side of (3.15) are of the form $\langle J\Phi\bar\delta\Phi\rangle$ while the right hand side of (3.14) behaves as $\langle\Phi\Phi\rangle\partial^2(\frac{1}{z^2})$. It is therefore necessary for the OPE to have the form $J\Phi\sim\delta\Phi$, so that terms like $\langle\delta\Phi\bar\delta\Phi\rangle=\langle[Q,\Phi]_\pm[\bar Q,\Phi]_\pm\rangle$ arise on the right hand side of (3.15). Using $\{\bar Q,Q\}\sim P_\mu$ such terms give derivatives, leading to $P_\mu\langle\Phi\Phi\rangle$. We need one more derivative in the OPE in order to match (3.14) which suggests (3.16). Also, $J\Phi\sim\bar\delta\Phi$ is ruled out by a mismatch of SU(4) indices. Finally, terms with higher numbers of δ and $\bar\delta$ operators have been dropped because they lead to less singular terms.

We have used the relation $[Q_{i\alpha},S_{\mu\nu}]=[\bar Q^{j\dot\alpha},S_{\mu\nu}]=0$ which can be proven as follows. the commutator $[Q_{i\alpha},S_{\mu\nu}]$ could contain symmetry generators in the $(1/2,0)$ or $(3/2,0)$ representations. However, there are no $(3/2,0)$ generators and the only possible $(1/2,0)$ generator is $Q_{i\alpha}$. The superconformal generator, $S^i{}_\alpha$, is in the $(1/2,0)$ representation but is ruled out by its $\mathrm{SU}(4)_R$ quantum numbers. So the most general possibility is

$$
[Q_{i\alpha},S_{\mu\nu}]=c_i{}^j(\sigma_{\mu\nu})_\alpha{}^\beta Q_{j\beta}\,, \tag{3.17}
$$

which, on conjugation, yields

$$
[\bar Q^{i\dot\alpha},S_{\mu\nu}]=c^i{}_j(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\bar Q^{j\dot\beta}\,, \tag{3.18}
$$

where $(c_i{}^j)^* = c^i{}_j$. Now consider the Jacobi identity

$$[S_{\mu\nu}, \{Q_{i\alpha}, \bar{Q}^{j\dot{\beta}}\}] + \{Q_{i\alpha}, [\bar{Q}^{j\dot{\beta}}, S_{\mu\nu}]\} - \{\bar{Q}^{j\dot{\beta}}, [S_{\mu\nu}, Q_{i\alpha}]\} = 0\,. \tag{3.19}$$

Using the relation $[S_{\mu\nu}, P_\lambda] = 0$, (3.19) gives

$$c_i{}^j(\sigma_{\mu\nu}\sigma_\lambda\epsilon)_\alpha{}^{\dot{\beta}} P^\lambda + c^j{}_i(\sigma_\lambda\epsilon)_\alpha{}^{\dot{\alpha}}(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}} P^\lambda = 0\,. \tag{3.20}$$

Contracting with $(\sigma_{\mu\nu})_\beta{}^\alpha$, equation (3.20) yields

$$c_i{}^j(\sigma_\mu\epsilon)_\beta{}^{\dot{\beta}} P^\mu = 0\,, \tag{3.21}$$

which implies that $c_i{}^j = 0$. The crucial step is that $[S_{\mu\nu}, P_\lambda] = 0$, i.e., momentum commutes with 'intrinsic' angular momentum. On the other hand, for the 'orbital' angular momentum, $L_{\mu\nu}$, we know that

$$[L_{\mu\nu}, P_\lambda] = i(\eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu)\,.$$

The Jacobi identity (3.19) for $L_{\mu\nu}$ implies that the structure constant, $\tilde{c}$, in $[Q_{i\alpha}, L_{\mu\nu}]$ satisfies $\tilde{c}_i{}^j = \delta_i{}^j$. This is exactly as it should be because using $M_{\mu\nu} = L_{\mu\nu} + iS_{\mu\nu}$, this gives

$$[Q_{i\alpha}, M_{\mu\nu}] = (\sigma_{\mu\nu})_\alpha{}^\beta Q_{i\beta}\,.$$

In summary, inserting (3.16) into (3.15), we obtain (to leading order) all terms in (3.14). This justifies the OPE (3.16). As a check, we note that from (3.16), we obtain the supercurrent Ward identity (2.27). Note that the $S_{\mu\nu}$ terms do not contribute to the Ward identity.

3.1.3 The $\delta^2\mathcal{O}_2$ OPE

Now given (3.16), we want to construct the OPE of $\delta^2\mathcal{O}_2$. So we need to peel off another $\bar{\delta}$. Restricting to the most singular terms, we first calculate the three-point function

$$\langle J_{\alpha\mu i}(z)\Phi^I(x)[\bar{Q}^j{}_{\dot{\alpha}}, \Phi^J(y)]_\pm\rangle\,. \tag{3.22}$$

Consider the contribution to the correlator as $z \to x$ and $z \to y$. To leading order, we see that

$$\begin{aligned}
&\langle J_{\alpha\mu i}(z)\Phi^I(x)[\bar{Q}^j{}_{\dot{\alpha}}, \Phi^J(y)]_\pm\rangle = \\
&\quad = \pm\delta^j{}_i(\sigma^\nu)_{\alpha\dot{\alpha}}\langle\Phi^I(x)\Phi^J(y)\rangle\partial_\mu\partial_\nu\left(\frac{1}{(z-x)^2} + \frac{1}{(z-y)^2}\right) \pm \\
&\quad\quad \pm\delta^j{}_i(\sigma_\mu)_{\alpha\dot{\alpha}}\langle\Phi^I(x)\Phi^J(y)\rangle\left(\delta^4(z-x) + \delta^4(z-y)\right) \pm \\
&\quad\quad \pm i\delta^j{}_i\langle\Phi^K(x)\Phi^J(y)\rangle\Big[S_{\mu\nu}\sigma_\lambda\partial^\lambda\partial^\nu + S^{\lambda\nu}\sigma_{\nu\mu}\sigma^\rho\partial_\lambda\partial_\rho + \\
&\quad\quad\quad + S^{\nu\rho}\sigma_\nu\partial_\rho\partial_\mu + S^{\rho\lambda}\sigma_\nu\bar{\sigma}_{\rho\lambda}\partial_\mu\partial^\nu\Big]^I_{\alpha\dot{\alpha}K}\frac{1}{(z-x)^2} \pm \\
&\quad\quad \pm i\delta^j{}_i\langle\Phi^I(x)\Phi^K(y)\rangle\Big[S_{\mu\nu}\sigma_\lambda\partial^\lambda\partial^\nu + S^{\lambda\nu}\sigma_{\nu\mu}\sigma^\rho\partial_\lambda\partial_\rho +
\end{aligned}$$

$$+ S^{\nu\rho}\sigma_\nu\partial_\rho\partial_\mu + S^{\rho\lambda}\sigma_\nu\bar{\sigma}_{\rho\lambda}\partial_\mu\partial^\nu\Big]^J_{\alpha\dot{\alpha}K}\frac{1}{(z-y)^2} \pm$$
$$\pm i\delta^j{}_i\langle\Phi^K(x)\Phi^J(y)\rangle[S_{\mu\nu}\sigma^\nu + S^{\rho\lambda}\sigma_\mu\bar{\sigma}_{\rho\lambda}]^I_{\alpha\dot{\alpha}K}\delta^4(z-x) \pm$$
$$\pm i\delta^j{}_i\langle\Phi^I(x)\Phi^K(y)\rangle[S_{\mu\nu}\sigma^\nu + S^{\rho\lambda}\sigma_\mu\bar{\sigma}_{\rho\lambda}]^J_{\alpha\dot{\alpha}K}\delta^4(z-y)\,. \tag{3.23}$$

We also need to make use of the relation

$$\{Q_j{}^\beta, [Q_{i\alpha}, Q^{ij}{}_{kl}]\} = \epsilon_{klmn}\Sigma_\alpha{}^{\beta mn}\,, \tag{3.24}$$

which follows from (3.3) and (3.4) and implies

$$[Q^{k\dot{\beta}}, \Sigma_\alpha^{\beta mn}]\epsilon_{ikmn}\sigma^\mu_{\beta\dot{\beta}} = J^\mu_{\alpha i}\,. \tag{3.25}$$

Using (3.11) and (3.25), the left hand side of (3.23) can be written as

$$\langle J^\mu_{\alpha i}(z)\Phi^I(x)[\bar{Q}^j{}_{\dot{\alpha}}, \Phi^J(y)]_\pm\rangle = \epsilon_{ikmn}(\sigma^\mu)_{\beta\dot{\beta}}\langle\Sigma_\alpha{}^{\beta mn}(z)\{\bar{Q}^{k\dot{\beta}}, \Phi^I(x)[\bar{Q}^j{}_{\dot{\alpha}}, \Phi^J(y)]_\pm\}\rangle\,. \tag{3.26}$$

Exactly along the lines of section 3.1.2, we make a preliminary ansatz of the form

$$\Sigma_\alpha{}^{\beta ij}(z)\Phi^I(x) \sim \frac{\epsilon^{ijkl}}{(z-x)^2}[Q_l{}^\beta, [Q_{k\alpha}, \Phi^I(x)]_\pm]_\mp +$$
$$+ i\frac{\epsilon^{ijkl}}{(z-x)^2}(S^{\nu\rho})^I{}_J(\sigma_{\mu\nu})_\alpha{}^\beta[Q_l{}^\gamma, [Q_{k\delta}, \Phi^J(x)]_\pm]_\mp(\sigma_\rho{}^\mu)_\gamma{}^\delta +$$
$$+\text{less singular terms}\,. \tag{3.27}$$

Using this ansatz, we can compute (3.26). There are terms of the form

$$\langle[Q, [Q, \Phi]], [\bar{Q}, [\bar{Q}, \Phi]]\rangle$$

which give rise to 2 derivative terms like $P^2\langle\Phi\Phi\rangle$. At the end, one obtains to leading order, the expression in (3.23).

If we did not have the $\Phi^I(x)\eta_{\mu\nu}\delta^4(z-x)$ contact term in (3.2), there would be no need for the second term in (3.16). However, in this case, the contact terms in (3.23) which do not involve $S_{\mu\nu}$ would not appear. Then the ansatz of (3.27) would not work. So this ansatz works consistently only when contact terms are correctly taken into acount. However, unlike the previous OPE expressions (3.2) and (3.16), (3.27) is incomplete. There should be more terms in the $\delta^2\mathcal{O}_2$ OPE (note from (3.3) and (3.4), we see that $\delta^2\mathcal{O}_2 \sim \Sigma$). Let us explain why this is the case.

From the definition of Σ in (3.8), we see that it contains two types of terms:

- $\epsilon_{\gamma\beta}\sigma_\alpha^{\mu\nu\gamma}B^{ij}_{\mu\nu}$, which is antisymmetric in ij and symmetric in $\alpha\beta$.
- $\epsilon_{\alpha\beta}\mathcal{E}^{ij}$ which is symmetric in ij and antisymmetric in $\alpha\beta$.

Clearly any term in the OPE of Σ with any Φ^I should be one of these two types. In fact, any term not of either form will not appear in the OPE. So schematically, we can write

$$\Sigma_\alpha{}^{\beta ij}\Phi \sim \sigma_\alpha^{\mu\nu\beta}\mathcal{P}^{[ij]}_{\mu\nu} + \delta_\alpha{}^\beta\mathcal{Q}^{(ij)}\,. \tag{3.28}$$

With this definition,

$$B^{ij}_{\mu\nu}\Phi \sim \mathcal{P}^{[ij]}_{\mu\nu} + \mathcal{K}^{[ij]}_{\mu\nu}\,, \tag{3.29}$$

where $\sigma^{\mu\nu}_{\alpha}{}^{\beta}\mathcal{K}^{[ij]}_{\mu\nu} = 0$ (for example, $\bar{\sigma}^{\dot{\alpha}}_{\mu\nu\,\dot{\beta}}\mathcal{Z}^{[ij]\dot{\beta}}{}_{\dot{\alpha}}$ is a possible term in $\mathcal{K}^{[ij]}_{\mu\nu}$). Also by definition

$$\mathcal{E}^{ij}\Phi \sim \mathcal{Q}^{(ij)}\,. \tag{3.30}$$

It is easy to see that all the terms on the right hand side of (3.27) take the form $(\sigma^{\mu\nu})_{\alpha}{}^{\beta}\mathcal{P}^{[ij]I}_{\mu\nu}$, where

$$\begin{aligned}\mathcal{P}^{[ij]I}_{\mu\nu} &= -\frac{1}{2}\frac{\epsilon^{ijkl}}{(z-x)^2}(\sigma_{\mu\nu})_{\alpha}{}^{\beta}[Q_l{}^{\alpha},[Q_{k\beta},\Phi^I(x)]_{\pm}]_{\mp}\\ &\quad -i\frac{\epsilon^{ijkl}}{(z-x)^2}(S_{\nu\rho})^I{}_J(\sigma_{\mu}{}^{\rho})_{\alpha}{}^{\beta}[Q_l{}^{\alpha},[Q_{k\beta},\Phi^J(x)]_{\pm}]_{\mp}\,.\end{aligned} \tag{3.31}$$

So all the terms that appear in (3.27) are terms in the OPE of $B^{ij}_{\mu\nu}$ with Φ^I. This is what we expect because (see (2.5)) under $\bar{\delta}$, $B^{ij}_{\mu\nu}$ transforms into $J^{\alpha}_{\mu i}$, while $\mathcal{E}^{ij}$ vanishes (modulo descendents). Schematically, $\bar{\delta}B \sim J$ and $\bar{\delta}\mathcal{E} \sim 0$. So on removing a $\bar{\delta}$ from $J^{\alpha}_{\mu i}$ in a correlator to obtain the terms in the Σ OPE, we should only obtain terms in the $B\Phi$ OPE since $\mathcal{E}$ is in the kernel of $\bar{\delta}$. The descendent term plays no role because it does not appear in our definition of the supercurrent (3.11).

From our prior analysis, we do not expect to see terms in the Σ OPE that arise from $\mathcal{E}$ so (3.27) is incomplete. In order to obtain these terms in the OPE of $\mathcal{E}^{ij}$ with Φ^I, we make use of the relations

$$\begin{aligned}[][Q_{k\alpha},\mathcal{E}^{ij}] = \delta_k{}^{(i}\Lambda^{j)}{}_{\alpha}\,,\qquad \{Q_j{}^{\beta},\Lambda^i{}_{\alpha}\} = i\delta_j{}^i\delta_{\alpha}{}^{\beta}\mathcal{O}_{\tau}\,,\\ [Q_i,\mathcal{O}_{\tau}] = \partial^{\mu}J_{\mu i} + 2\sigma^{\mu}\bar{\sigma}^{\nu}\partial_{\mu}J_{\nu i}\,,\end{aligned} \tag{3.32}$$

(which follow from from (2.5)) to show that

$$\partial_{\mu}J^{\mu}_{k\beta} + 2(\sigma^{\mu}\bar{\sigma}^{\nu}\partial_{\mu}J_{\nu k})_{\beta} = [Q_{k\alpha},\{Q_j{}^{\beta},[Q_{i\beta},\mathcal{E}^{ij}]\}]\,. \tag{3.33}$$

We have set various numerical factors to one in this relation since their values will not be relevant in the following analysis. Schematically, (3.33) has the form $\delta^3\mathcal{E} \sim \partial J$ so $\mathcal{E}$ is related to a current.

Using (3.33), we consider the equality

$$\begin{aligned}&\partial_{\mu}\langle J^{\mu}_{i\alpha}(z)\Phi^I(x)[\bar{Q}^j{}_{\dot{\alpha}},\Phi^J(y)]_{\pm}\rangle + 2(\sigma_{\mu}\bar{\sigma}_{\nu})_{\alpha}{}^{\beta}\partial^{\mu}\langle J^{\nu}_{i\beta}(z)\Phi^I(x)[\bar{Q}^j{}_{\dot{\alpha}},\Phi^J(y)]_{\pm}\rangle\\ &\qquad = \langle\mathcal{E}^{kl}(z)\{Q_{k\beta},[Q_l{}^{\beta},\{Q_{i\alpha},\Phi^I(x)[\bar{Q}^j{}_{\dot{\alpha}},\Phi^J(y)]_{\pm}\}]\}\rangle.\end{aligned} \tag{3.34}$$

We will first evaluate the contribution to the correlators on the left-hand side of (3.34). Using the Ward identities (2.27) and (2.30), we find

$$\begin{aligned}&\partial_{\mu}\langle J^{\mu}_{i\alpha}(z)\Phi^I(x)[\bar{Q}^j{}_{\dot{\alpha}},\Phi^J(y)]_{\pm}\rangle + 2(\sigma_{\mu}\bar{\sigma}_{\nu})_{\alpha}{}^{\beta}\partial^{\mu}\langle J^{\nu}_{i\beta}(z)\Phi^I(x)[\bar{Q}^j{}_{\dot{\alpha}},\Phi^J(y)]_{\pm}\rangle =\\ &\quad = \delta^4(z-x)\langle[Q_{i\alpha},\Phi^I(x)]_{\pm}[\bar{Q}^j{}_{\dot{\alpha}},\Phi^J(y)]_{\pm}\rangle + \delta^4(z-y)\langle\Phi^I(x)[Q_{i\alpha},[\bar{Q}^j{}_{\dot{\alpha}},\Phi^J(y)]_{\pm}]_{\mp}\rangle +\\ &\quad\quad +\partial^z_{\alpha\dot{\alpha}}\delta^4(z-x)\langle[\bar{S}^{\dot{\alpha}}_i - x^{\alpha\dot{\alpha}}Q_{i\alpha},\Phi^I(x)]_{\pm}[\bar{Q}^j{}_{\dot{\alpha}},\Phi^J(y)]_{\pm}\rangle +\end{aligned}$$

$$+\partial^z_{\alpha\dot\alpha}\delta^4(z-y)\langle\Phi^I(x)[\bar S^{\dot\alpha}_i - y^{\alpha\dot\alpha}Q_{i\alpha},[\bar Q^j{}_{\dot\alpha},\Phi^J(y)]_\pm]_\mp\rangle\,. \tag{3.35}$$

Eventually (from (1.4)) the insertion point of $\mathcal{O}_\tau$ is to be integrated over z. This simplifies the possible terms that need to be considered in the OPE. Using

$$\int d^4z\partial^z_\mu\delta^4(z-x)f(x)=0\,, \tag{3.36}$$

we see that terms involving derivative(s) acting on delta functions can be ignored. Hence, to leading order the terms that will be of relevance are given by

$$\begin{aligned}\partial_\mu\langle J^\mu_{i\alpha}(z)\Phi^I(x)[\bar Q^j{}_{\dot\alpha},\Phi^J(y)]_\pm\rangle + 2(\sigma_\mu\bar\sigma_\nu)_\alpha{}^\beta\partial^\mu\langle J^\nu_{i\beta}(z)\Phi^I(x)[\bar Q^j{}_{\dot\alpha},\Phi^J(y)]_\pm\rangle \sim\\ \sim[\delta^4(z-x)+\delta^4(z-y)]\partial_{\alpha\dot\alpha}\langle\Phi^I(x)\Phi^J(y)\rangle\,. \end{aligned}\tag{3.37}$$

Note that all the contributions come from the $\partial^\mu J_{\mu i}$ term only.

Next we need to evaluate the right hand side of (3.34) as $z\to x$ and $z\to y$, retaining the leading order terms. We make the ansatz

$$\mathcal{E}^{ij}(z)\Phi^I(x)\sim\frac{1}{(z-x)^2}[\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\alpha},\Phi^I(x)]_\pm]_\mp + \text{less singular terms}\,, \tag{3.38}$$

where the term on the right hand side is symmetric in ij by construction. Inserting this ansatz into the right hand side of (3.34), we see from (3.35) that the equality holds to this order. This OPE does not involve any $S_{\mu\nu}$ terms at this order. So we can write the complete OPE as

$$\begin{aligned}\Sigma_\alpha{}^{\beta ij}(z)\Phi^I(x)\sim\ &\frac{\epsilon^{ijkl}}{(z-x)^2}[Q_l{}^\beta,[Q_{k\alpha},\Phi^I(x)]_\pm]_\mp + \frac{\delta_\alpha{}^\beta}{(z-x)^2}[\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\alpha},\Phi^I(x)]_\pm]_\mp +\\ &+i\frac{\epsilon^{ijkl}}{(z-x)^2}(S^{\nu\rho})^I{}_J(\sigma_{\mu\nu})_\alpha{}^\beta[Q_l{}^\gamma,[Q_{k\delta},\Phi^J(x)]_\pm]_\mp(\sigma_\rho{}^\mu)_\gamma{}^\delta +\\ &+\text{less singular terms}\,. \end{aligned}\tag{3.39}$$

It is important for our purposes to construct all the less singular terms in (3.39). This is because we will use the $\delta^2\mathcal{O}_2$ OPE to recursively construct the $\mathcal{O}_\tau$ OPE which will have singular terms of different orders. All such terms can contribute to the integral (1.4). Also, these terms will be needed when we check the consistency of the $\delta^2\mathcal{O}_2$ OPE. Since these terms involve lengthy expressions, we present them in appendix C. The complete result for the $\delta^2\mathcal{O}_2$ OPE plays an important role in our subsequent analysis, so we have also provided a number of checks on the OPE in appendix D.

3.2 The OPE for Λ and $\mathcal{O}_\tau$

The final steps require us to determine the OPE structure for $\delta^3\mathcal{O}_2\equiv\Lambda$ and then $\mathcal{O}_\tau\equiv\delta^4\mathcal{O}_2$ from the $\delta^2\mathcal{O}_2$ OPE. We will first express $\mathcal{O}_\tau$ in terms of the supercurrents and $\mathcal{O}_2$ using the relations

$$[Q_{k\gamma},\Sigma_\alpha{}^{\beta ij}]=\epsilon_{\alpha\gamma}\delta_k{}^{[i}\Lambda^{j]\beta}+\delta_\gamma{}^\beta\delta_k{}^{[i}\Lambda^{j]}{}_\alpha+\delta_\alpha{}^\beta\delta_k{}^{(i}\Lambda^{j)}{}_\gamma\,,\qquad \{Q_j{}^\beta,\Lambda^i{}_\alpha\}=i\delta_j{}^i\delta_\alpha{}^\beta\mathcal{O}_\tau\,, \tag{3.40}$$

which can be deduced from (2.5)

The supervariations of all the operators in the current multiplet depend only on the combination Σ and not on B or $\mathcal{E}$ individually. Very schematically, from (2.5), we see that

$$\hat{\delta}\chi \sim \Sigma\eta + \cdots\,, \qquad \hat{\delta}J \sim \partial\Sigma\eta + \cdots\,, \qquad \hat{\delta}\Lambda \sim \partial\Sigma\bar{\eta} + \cdots\,. \tag{3.41}$$

So $\mathcal{O}_\tau$ can be expressed as δ^2 acting on Σ. Explicitly, using the supersymmetry transformations leads to (dropping overall numerical coefficients)

$$B^{ij}_{\mu\nu} = \frac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta \epsilon^{ijkl}\{Q_n{}^\alpha, [Q_{m\beta}, Q^{mn}{}_{kl}]\}\,, \qquad \mathcal{E}^{ij} = \epsilon^{jlmn}\{Q_l{}^\alpha, [Q_{k\alpha}, Q^{ki}{}_{mn}]\}\,, \tag{3.42}$$

so that

$$\Sigma_\alpha{}^{\beta ij} = \epsilon^{ijkl}\{Q_{m\alpha}, [Q_n{}^\beta, Q^{mn}{}_{kl}]\} + \delta_\alpha{}^\beta \epsilon^{jklm}\{Q_k{}^\gamma, [Q_{n\gamma}, Q^{ni}{}_{lm}]\}\,. \tag{3.43}$$

Using (3.40), we see that

$$\Lambda^i_\alpha = [Q_{j\beta}, \Sigma_\alpha{}^{\beta ij}]\,, \tag{3.44}$$

and finally we arrive at the desired relation between $\mathcal{O}_\tau$ and Σ,

$$\mathcal{O}_\tau = \{Q_i{}^\alpha, [Q_{j\beta}, \Sigma_\alpha{}^{\beta ij}]\}\,. \tag{3.45}$$

Now consider the leading terms (3.39) in the Σ OPE. Acting with $Q_{j\beta}$ on (3.39) and using (3.44), we find that

$$\begin{aligned}
\Lambda^i_\alpha(z)\Phi^I(x) + \Sigma_\alpha{}^{\beta ij}(z)[Q_{j\beta}, \Phi^I(x)]_\pm \sim\; & \frac{1}{(z-x)^2}[Q_{j\alpha}, [\bar{Q}^i{}_{\dot\alpha}, [\bar{Q}^{j\dot\alpha}, \Phi^I(x)]_\pm]_\mp]_\pm + \\
& + i\frac{\epsilon^{ijkl}}{(z-x)^2}(S^{\nu\rho})^I{}_J(\sigma_{\mu\nu})_\alpha{}^\beta \times \\
& \times [Q_{j\beta}, [Q_l{}^\gamma, [Q_{k\delta}, \Phi^J(x)]_\pm]_\mp]_\pm (\sigma_\rho{}^\mu)_\gamma{}^\delta + \cdots\,.
\end{aligned} \tag{3.46}$$

Using (3.39) once more in the second term on the left hand side of (3.46) gives the $\delta^3\mathcal{O}_2$ OPE

$$\begin{aligned}
\Lambda^i_\alpha(z)\Phi^I(x) \sim\; & \frac{1}{(z-x)^2}[Q_{j\alpha}, [\bar{Q}^i{}_{\dot\alpha}, [\bar{Q}^{j\dot\alpha}, \Phi^I(x)]_\pm]_\mp]_\pm + \\
& + \frac{1}{(z-x)^2}[\bar{Q}^i{}_{\dot\alpha}, [\bar{Q}^{j\dot\alpha}, [Q_{j\alpha}, \Phi^I(x)]_\pm]_\mp]_\pm + \\
& + i\frac{\epsilon^{ijkl}}{(z-x)^2}(S^{\mu\nu})^I{}_J(\sigma_{\mu\nu})_\beta{}^\gamma [Q_{j\alpha}, [Q_l{}^\beta, [Q_{k\gamma}, \Phi^J(x)]_\pm]_\mp]_\pm + \cdots\,.
\end{aligned} \tag{3.47}$$

Acting on (3.47) with $Q_i{}^\alpha$ and using (3.45), gives the relation

$$\begin{aligned}
\mathcal{O}_\tau(z)\Phi^I(x) + \Lambda^i_\alpha(z)[Q_i{}^\alpha, \Phi^I(x)]_\pm \sim\; & \frac{1}{(z-x)^2}[Q_i{}^\alpha, [Q_{j\alpha}, [\bar{Q}^i{}_{\dot\alpha}, [\bar{Q}^{j\dot\alpha}, \Phi^I(x)]_\pm]_\mp]_\pm]_\mp + \\
& + \frac{1}{(z-x)^2}[Q_i{}^\alpha, [\bar{Q}^i{}_{\dot\alpha}, [\bar{Q}^{j\dot\alpha}, [Q_{j\alpha}, \Phi^I(x)]_\pm]_\mp]_\pm]_\mp + \cdots\,.
\end{aligned} \tag{3.48}$$

On using (3.47) in the second term on the left hand side of (3.48), we arrive at the most singular terms in the $\mathcal{O}_\tau$ OPE

$$\mathcal{O}_\tau(z)\Phi^I(x) \sim \frac{a_1}{(z-x)^2}[Q_i{}^\alpha, [Q_{j\alpha}, [\bar{Q}^i{}_{\dot\alpha}, [\bar{Q}^{j\dot\alpha}, \Phi^I(x)]_\pm]_\mp]_\pm]_\mp +$$

$$+\frac{a_2}{(z-x)^2}[Q_i{}^\alpha,[\bar{Q}^i{}_{\dot\alpha},[\bar{Q}^{j\dot\alpha},[Q_{j\alpha},\Phi^I(x)]_\pm]_\mp]_\pm]_\mp+$$
$$+\frac{a_3}{(z-x)^2}[\bar{Q}^i{}_{\dot\alpha},[\bar{Q}^{j\dot\alpha},[Q_i{}^\alpha,[Q_{j\alpha},\Phi^I(x)]_\pm]_\mp]_\pm]_\mp+\cdots\,. \tag{3.49}$$

The coefficients, a_i, are generally functions of $(\tau,\bar\tau)$. Note also that the contributions considered so far to the $\mathcal{O}_\tau$ OPE (3.49) are independent of $S_{\mu\nu}$.

Some terms in (3.47) and (3.49) can be re-ordered at the expense of introducing derivatives. This only lengthens the expressions so we will keep the displayed ordering. To illustrate this point, note that we can re-order (3.49) to give (ignoring the a_i coefficients)

$$\begin{aligned}\mathcal{O}_\tau(z)\Phi^I(x)\sim&\left(-4\pi^2\delta^4(z-x)+\frac{1}{(z-x)^2}\partial^2+2\partial^\mu\frac{1}{(z-x)^2}\partial_\mu\right)\Phi^I(x)+\\&+\partial_{\alpha\dot\alpha}\left(\frac{1}{(z-x)^2}[Q_i{}^\alpha,[\bar{Q}^{i\dot\alpha},\Phi^I(x)]_\pm]_\mp\right)+\\&+\frac{1}{(z-x)^2}[Q_i{}^\alpha,[Q_{j\alpha},[\bar{Q}^i{}_{\dot\alpha},[\bar{Q}^{j\dot\alpha},\Phi^I(x)]_\pm]_\mp]_\pm]_\mp+\cdots\,.\end{aligned}$$

With this ordering, the leading singularity in the $\mathcal{O}_\tau$ OPE appears to be a contact term rather than a power law singularity. Indeed, if we were to permit integration by parts (allowed, for example, when we integrate over z), all the leading singular terms can be rewritten as contact terms.

The remaining contributions to the Λ and $\mathcal{O}_\tau$ OPE can be deduced in a straightforward way from the expressions in (C.3), (C.6), (C.8) and (C.9). However, for simplicity, we will restrict to Φ^I which are Lorentz scalars transforming in any $\mathrm{SU}(4)_R$ representation. This simplification means that we can set all $S_{\mu\nu}$ terms to zero. It is a tedious, but straightforward, exercise to determine these terms should they be needed (we will require these terms in [19]).

From the terms (C.3) in the Σ OPE, it follows that the Λ OPE gets contributions

$$\begin{aligned}\Lambda^i_\alpha(z)\Phi(x)\sim&\frac{r^\mu}{r^2}\partial_\mu[Q_{j\alpha},\{\bar{Q}^i{}_{\dot\alpha},[\bar{Q}^{j\dot\alpha},\Phi(x)]\}]+\frac{r^\mu}{r^2}\partial_\mu[\bar{Q}^i{}_{\dot\alpha},\{\bar{Q}^{j\dot\alpha},[Q_{j\alpha},\Phi(x)]\}]+\\&+(\sigma_{\mu\nu})_\alpha{}^\beta(\bar\sigma^{\mu\rho})^{\dot\alpha}{}_{\dot\beta}\frac{r^\nu}{r^2}\partial_\rho[Q_{j\beta},\{\bar{Q}^i{}_{\dot\alpha},[\bar{Q}^{j\dot\beta},\Phi(x)]\}]+\\&+(\sigma_{\mu\nu})_\alpha{}^\beta(\bar\sigma^{\mu\rho})^{\dot\alpha}{}_{\dot\beta}\frac{r^\nu}{r^2}\partial_\rho[\bar{Q}^i{}_{\dot\alpha},\{\bar{Q}^{j\dot\beta},[Q_{j\beta},\Phi(x)]\}]\,,\end{aligned} \tag{3.50}$$

(dropping coupling constant dependent coefficients) while the $\mathcal{O}_\tau$ OPE gets contributions

$$\begin{aligned}\mathcal{O}_\tau(z)\Phi(x)\sim&\,a_4\frac{r^\mu}{r^2}\partial_\mu\{Q_i{}^\alpha,[Q_{j\alpha},\{\bar{Q}^i{}_{\dot\alpha},[\bar{Q}^{j\dot\alpha},\Phi(x)]\}]\}+\\&+a_5\frac{r^\mu}{r^2}\partial_\mu\{Q_i{}^\alpha,[\bar{Q}^i{}_{\dot\alpha},\{\bar{Q}^{j\dot\alpha},[Q_{j\alpha},\Phi(x)]\}]\}+\\&+a_6\frac{r^\mu}{r^2}\partial_\mu\{\bar{Q}^i{}_{\dot\alpha},[\bar{Q}^{j\dot\alpha},\{Q_i{}^\alpha,[Q_{j\alpha},\Phi(x)]\}]\}+\\&+a_7(\sigma_{\mu\nu})_\alpha{}^\beta(\bar\sigma^{\mu\rho})^{\dot\alpha}{}_{\dot\beta}\frac{r^\nu}{r^2}\partial_\rho\{Q_i{}^\alpha,[Q_{j\beta},\{\bar{Q}^i{}_{\dot\alpha},[\bar{Q}^{j\dot\beta},\Phi(x)]\}]\}+\\&+a_8(\sigma_{\mu\nu})_\alpha{}^\beta(\bar\sigma^{\mu\rho})^{\dot\alpha}{}_{\dot\beta}\frac{r^\nu}{r^2}\partial_\rho\{Q_i{}^\alpha,[\bar{Q}^i{}_{\dot\alpha},\{\bar{Q}^{j\dot\beta},[Q_{j\beta},\Phi(x)]\}]\}+\end{aligned}$$

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$$+a_9(\sigma_{\mu\nu})_\alpha{}^\beta(\bar\sigma^{\mu\rho})^{\dot\alpha}{}_{\dot\beta}\frac{r^\nu}{r^2}\partial_\rho\{\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\beta},\{Q_i{}^\alpha,[Q_{j\beta},\Phi(x)]\}]\}\,, \tag{3.51}$$

where $a_4,\ldots,a_9$ are undetermined functions of the coupling.

It is easy to see that there are no terms in the OPE of Λ or $\mathcal{O}_\tau$ with a Lorentz scalar from (C.6). However, more contributions arise by considering (C.8) which gives

$$\begin{aligned}\Lambda^i_\alpha(z)\Phi(x)\sim{}&\frac{r^\mu}{r^2}(\sigma_\mu)_{\beta\dot\beta}[Q_{j\alpha},\{\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\alpha},\{\bar Q^{k\dot\beta},[Q_k{}^\beta,\Phi(x)]\}]\}]+\\&+\frac{r^\mu}{r^2}(\sigma_\mu)_{\beta\dot\beta}[\bar Q^i{}_{\dot\alpha},\{\bar Q^{j\dot\alpha},[\bar Q^{k\dot\beta},\{Q_k{}^\beta,[Q_{j\alpha},\Phi(x)]\}]\}]+\\&+(\sigma_{\mu\nu})_\alpha{}^\beta\frac{r_\lambda}{r^2}(\sigma^\nu)_{\gamma\dot\gamma}(\bar\sigma^{\mu\lambda})^{\dot\alpha}{}_{\dot\beta}[Q_{j\beta},\{\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\beta},\{\bar Q^{k\dot\gamma},[Q_k{}^\gamma,\Phi(x)]\}]\}]+\\&+(\sigma_{\mu\nu})_\alpha{}^\beta\frac{r_\lambda}{r^2}(\sigma^\nu)_{\gamma\dot\gamma}(\bar\sigma^{\mu\lambda})^{\dot\alpha}{}_{\dot\beta}[\bar Q^i{}_{\dot\alpha},\{\bar Q^{j\dot\beta},[\bar Q^{k\dot\gamma},\{Q_k{}^\gamma,[Q_{j\beta},\Phi(x)]\}]\}]\end{aligned} \tag{3.52}$$

and

$$\begin{aligned}\mathcal{O}_\tau(z)\Phi(x)\sim{}&a_{10}\frac{r^\mu}{r^2}(\sigma_\mu)_{\beta\dot\beta}\{Q_i{}^\alpha,[Q_{j\alpha},\{\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\alpha},\{\bar Q^{k\dot\beta},[Q_k{}^\beta,\Phi(x)]\}]\}]\}+\\&+a_{11}\frac{r^\mu}{r^2}(\sigma_\mu)_{\beta\dot\beta}\{Q_i{}^\alpha,[\bar Q^i{}_{\dot\alpha},\{\bar Q^{j\dot\alpha},[\bar Q^{k\dot\beta},\{Q_k{}^\beta,[Q_{j\alpha},\Phi(x)]\}]\}]\}+\\&+a_{12}\frac{r^\mu}{r^2}(\sigma_\mu)_{\beta\dot\beta}\{\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\alpha},\{\bar Q^{k\dot\beta},[Q_k{}^\beta,\{Q_i{}^\alpha,[Q_{j\alpha},\Phi(x)]\}]\}]\}+\\&+a_{13}(\sigma_{\mu\nu})_\alpha{}^\beta\frac{r_\lambda}{r^2}(\sigma^\nu)_{\gamma\dot\gamma}(\bar\sigma^{\mu\lambda})^{\dot\alpha}{}_{\dot\beta}\{Q_i{}^\alpha,[Q_{j\beta},\{\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\beta},\{\bar Q^{k\dot\gamma},[Q_k{}^\gamma,\Phi(x)]\}]\}]\}+\\&+a_{14}(\sigma_{\mu\nu})_\alpha{}^\beta\frac{r_\lambda}{r^2}(\sigma^\nu)_{\gamma\dot\gamma}(\bar\sigma^{\mu\lambda})^{\dot\alpha}{}_{\dot\beta}\{Q_i{}^\alpha,[\bar Q^i{}_{\dot\alpha},\{\bar Q^{j\dot\beta},[\bar Q^{k\dot\gamma},\{Q_k{}^\gamma,[Q_{j\beta},\Phi(x)]\}]\}]\}+\\&+a_{15}(\sigma_{\mu\nu})_\alpha{}^\beta\frac{r_\lambda}{r^2}(\sigma^\nu)_{\gamma\dot\gamma}(\bar\sigma^{\mu\lambda})^{\dot\alpha}{}_{\dot\beta}\{\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\beta},\{\bar Q^{k\dot\gamma},[Q_k{}^\gamma,\{Q_i{}^\alpha,[Q_{j\beta},\Phi(x)]\}]\}]\}\,,\end{aligned} \tag{3.53}$$

where $a_{10},\ldots,a_{15}$ are further undetermined functions of the coupling. Finally, considering the contribution from (C.9), we find

$$\Lambda^i_\alpha(z)\Phi(x)\sim\epsilon^{ijkl}\frac{r^\mu}{r^2}(\sigma_{\mu\nu})_\beta{}^\gamma\partial^\nu[Q_{j\alpha},\{Q_l{}^\beta,[Q_{k\gamma},\Phi(x)]\}]\,, \tag{3.54}$$

while there is no corresponding contribution to the $\mathcal{O}_\tau$ OPE.

In summary, the complete $\mathcal{O}_\tau$ OPE with a scalar Φ (modulo operators in different supermultiplets) is given by the sum of the expressions (3.49), (3.51) and (3.53). The coefficients $(a_1,\ldots,a_{15})$ are generally undetermined functions of $(\tau,\bar\tau)$.

4. Some properties of correlation functions

4.1 Operator normalization and contact terms

Now that we have established the form of the $\mathcal{O}_\tau$ OPE, we turn to the structure of correlation functions of local operators,

$$\langle\mathcal{O}^{I_1}(x_1)\cdots\mathcal{O}^{I_n}(x_n)\rangle\,. \tag{4.1}$$

We will always consider correlators of operators at separated points. Each correlation function transforms in some representation of $\mathrm{SL}(2,\mathbb{Z})$. This need not be a singlet representation, as shown for correlators involving the Konishi supermultiplet [30]. However, a correlator of BPS operators should map back to itself since each BPS operator is uniquely specified by its quantum numbers. Such a correlator should transform in a singlet representation of $\mathrm{SL}(2,\mathbb{Z})$ with fixed weights $(w,\bar{w})$. Under an $\mathrm{SL}(2,\mathbb{Z})$ transformation,

$$\tau \to \left(\frac{a\tau+b}{c\tau+d}\right), \qquad a,b,c,d \in \mathbb{Z} \tag{4.2}$$

a modular form, $\Theta^{(w,\bar{w})}$, in a singlet representation transforms in the following way:

$$\Theta^{(w,\bar{w})}(\tau,\bar{\tau}) \to (c\tau+d)^{w}(c\bar{\tau}+d)^{\bar{w}}\Theta^{(w,\bar{w})}(\tau,\bar{\tau})\,. \tag{4.3}$$

The weights we assign to BPS operators should be correlated with their $\mathrm{U}(1)_Y$ transformation properties in a way described in [4].

The $\mathrm{SL}(2,\mathbb{Z})$ transformation properties of a correlation function are important because the expression,

$$\frac{\partial}{\partial\tau}\left\langle \prod_r \mathcal{O}^{I_r}(x_r)\right\rangle = \left\langle \frac{\partial}{\partial\tau}\left(\prod_i \mathcal{O}^{I_i}(x_i)\right)\right\rangle + \frac{i}{4\tau_2}\int d^4z \left\langle \mathcal{O}_\tau(z)\prod_r \mathcal{O}^{I_r}(x_r)\right\rangle, \tag{4.4}$$

is not modular covariant at first sight. For the moment, let us restrict to correlators transforming in singlet representations of $\mathrm{SL}(2,\mathbb{Z})$. Let us examine the explicit τ derivative acting on each $\mathcal{O}^{I_i}$,

$$\frac{\partial}{\partial\tau}\mathcal{O}^{I_i}\,. \tag{4.5}$$

The operator (4.5) has the same quantum numbers as O^{I_i}. In particular, the conformal dimensions agree. If there is no degeneracy for operators with conformal dimension Δ_i then (4.5) must be proportional to O^{I_i}. This is the case for short operators. If there is a finite-dimensional degeneracy then we can again choose a basis of operators for which this statement is true. This leaves us with the freedom to rescale each $\mathcal{O}^{I_i}$ by a function of $(\tau,\bar{\tau})$. We can conclude that in this basis,

$$\frac{\partial}{\partial\tau}\mathcal{O}^{I_i} = \frac{i\alpha_{I_i}(\tau,\bar{\tau})}{\tau_2}\mathcal{O}^{I_i}\,. \tag{4.6}$$

Using our final rescaling degree of freedom, we choose to set α_{I_i} to a constant. For the current multiplet, the normalization of all operators is determined by the definition of $\mathcal{O}_\tau$ in terms of the action (1.5) together with the (coupling independent) supersymmetry transformations.

Equation (4.4) must be $\mathrm{SL}(2,\mathbb{Z})$ covariant if the correlator transforms in a singlet representation. It must therefore be the case that summing the tree-level contact terms between $\mathcal{O}_\tau$ and $\mathcal{O}^{I_i}$ together with the α_{I_i} from (4.6) has the net effect of replacing

$$i\tau_2\frac{\partial}{\partial\tau} \to D_w\,, \qquad -i\tau_2\frac{\partial}{\partial\bar{\tau}} \to D_{\bar{w}}\,.$$

We have defined modular covariant derivatives,

$$D_w = i\left(\tau_2\frac{\partial}{\partial\tau} - \frac{iw}{2}\right), \qquad \bar{D}_{\bar{w}} = -i\left(\tau_2\frac{\partial}{\partial\bar{\tau}} + \frac{i\bar{w}}{2}\right), \tag{4.7}$$

where $(w, \bar{w})$ is the weight of the correlation function. These issues have been discussed in [31].

To see whether this actually happens requires a computation of the tree-level contact term between $\mathcal{O}_\tau$ and each $\mathcal{O}^{I_i}$. However, in perturbation theory, precisely this piece of the contact term is scheme-dependent [32]. However, the full non-perturbative theory must require the particular choice consistent with $\mathrm{SL}(2,\mathbb{Z})$. We will assume that this contact term takes the value required by $\mathrm{SL}(2,\mathbb{Z})$ for long operators. Fortunately, for a BPS superconformal primary, we will be able to determine the weight of the operator (normalized in a particular way) by direct arguments without recourse to any duality assumptions.

We now use the $\mathcal{O}_\tau$ OPE given by (3.49), (3.51) and (3.53) to analyze the coupling dependence of correlators using (4.4). It is difficult to control the right hand side of (4.4) except for low-point functions where we know the exact space-time dependence of the correlator. We will consider those special cases in a moment.

Let us focus on the contribution to the right hand side of (4.4) from $z \to x_i$. This contribution is dominated by the singular terms in the OPE between $\mathcal{O}_\tau$ and $\mathcal{O}^{I_i}$. The contribution to the integral from z far from x_i is also present but will not be explicitly displayed in the following equation. In other words, we rewrite (4.4) in the form

$$\begin{aligned}\frac{\partial}{\partial \tau}\left\langle \prod_i \mathcal{O}^{I_i}(x_i)\right\rangle = & \left\langle \frac{\partial}{\partial\tau}\left(\prod_i \mathcal{O}^{I_i}(x_i)\right)\right\rangle + \\ & + \frac{i}{4\tau_2}\sum_j \int_{B^\epsilon_{x_j}} d^4z \left\langle \mathcal{O}_\tau(z)\prod_i \mathcal{O}^{I_i}(x_i)\right\rangle + \cdots ,\end{aligned} \tag{4.8}$$

where $B^\epsilon_{x_j}$ is a ball of radius ϵ surrounding the point x_j. The omitted terms refer to the contribution to the integral from the region outside each ball.

The right hand side of (4.8) is a sum of integrated $(n+1)$-point functions. In an approximation where we neglect the contribution from outside the balls, we can use the $\mathcal{O}_\tau$ OPE to replace each $(n+1)$-point function by an n-point function. In cases where we know something about the space-time dependence of the correlator, this approximation is sufficient to teach us something about the coupling dependence. This is essentially what we anticipated in the introduction. It will be interesting to see what information can be gained by using this relation in conjunction with recent results about the space-time dependence of 4-point functions [33, 34, 35].

4.2 Implications for two-point functions

We consider equation (4.8) restricted to two operator insertions. This analysis will constrain combinations of the a_i coefficients defined in section 3.2. This is also a good warm-up for higher point functions. To simplify the $\mathcal{O}_\tau$ OPE, let us take the inserted operators to be space-time scalars.

$$\frac{\partial}{\partial\tau}\langle\bar{\mathcal{O}}(x)\mathcal{O}(y)\rangle = \left\langle \frac{\partial}{\partial\tau}\left(\bar{\mathcal{O}}(x)\mathcal{O}(y)\right)\right\rangle + \frac{i}{4\tau_2}\int_{B^\epsilon_x} d^4z \langle\mathcal{O}_\tau(z)\bar{\mathcal{O}}(x)\mathcal{O}(y)\rangle +$$

$$+\frac{i}{4\tau_2}\int_{B_y^\epsilon} d^4z\langle\mathcal{O}_\tau(z)\bar{\mathcal{O}}(x)\mathcal{O}(y)\rangle+\cdots\,. \qquad (4.9)$$

Also, we will consider cases where the operator $\mathcal{O}$ and its superconformal descendents, $\delta\mathcal{O}$ and, $\delta^2\mathcal{O}$, are conformal primaries. This is automatically the case if $\mathcal{O}$ is a superconformal primary; in this case, $\bar{\mathcal{O}}=\mathcal{O}$. For more general situations, this restriction rules out the possibility $\mathcal{O}=\bar{\delta}\Upsilon$ because then $\delta\mathcal{O}\sim\partial_\mu\Upsilon$ is a conformal descendent. These simplifications will enable us to calculate the correlation functions appearing on the right hand side of (4.9) with more straightforward algebra.

We need to first determine the normalization of certain two-point functions. For any conformal primary Φ,

$$\langle\bar{\Phi}(x)\,\bar{\delta}\delta\,\Phi(y)\rangle=\langle\bar{\Phi}(x)\bar{Q}^j{}_{\dot{\alpha}}Q_{j\alpha}\Phi(y)\rangle=4i\partial^x_{\alpha\dot{\alpha}}\langle\bar{\Phi}(x)\Phi(y)\rangle\,. \qquad (4.10)$$

This follows from conformal invariance; if Φ has conformal dimension Δ then $\bar{\delta}\delta\Phi$ has dimension $\Delta+1$ so the correlator $\langle\bar{\Phi}(x)\,\bar{\delta}\delta\,\Phi(y)\rangle$ vanishes if $\bar{\delta}\delta\Phi$ is a conformal primary. We want to determine the exact coefficient appearing in this relation. Using the supersymmetry algebra, we find that

$$\langle[Q_{i\alpha},[\bar{Q}^j{}_{\dot{\alpha}},\bar{\Phi}(x)]_\pm]_\mp\Phi(y)\rangle=2\delta_i{}^jP^x_{\alpha\dot{\alpha}}\langle\bar{\Phi}(x)\Phi(y)\rangle+\langle\bar{\Phi}(x)[Q_{i\alpha},[\bar{Q}^j{}_{\dot{\alpha}},\Phi(y)]_\pm]_\mp\rangle\,. \qquad (4.11)$$

We also observe that

$$\begin{aligned}\langle\bar{\Phi}(x)[Q_{i\alpha},[\bar{Q}^j{}_{\dot{\alpha}},\Phi(y)]_\pm]_\mp\rangle &= \mp\langle[Q_{i\alpha},[\bar{Q}^j{}_{\dot{\alpha}},\Phi(y)]_\pm]_\mp\bar{\Phi}(x)\rangle=-\langle[Q_{i\alpha},[\bar{Q}^j{}_{\dot{\alpha}},\Phi(x)]_\pm]_\mp\bar{\Phi}(y)\rangle\\ &= -\langle[Q_{i\alpha},[\bar{Q}^j{}_{\dot{\alpha}},\bar{\Phi}(x)]_\pm]_\mp\Phi(y)\rangle\,,\end{aligned} \qquad (4.12)$$

which when substituted in (4.11) yields the relation

$$\langle[Q_{i\alpha},[\bar{Q}^j{}_{\dot{\alpha}},\bar{\Phi}(x)]_\pm]_\mp\Phi(y)\rangle=\delta_i{}^jP^x_{\alpha\dot{\alpha}}\langle\bar{\Phi}(x)\Phi(y)\rangle\,. \qquad (4.13)$$

Alternatively, one can deduce (4.13) in the following way: from the general arguments above, we know that

$$\langle[Q_{i\alpha},[\bar{Q}^j{}_{\dot{\alpha}},\bar{\Phi}(x)]_\pm]_\mp\Phi(y)\rangle=\rho\delta_i{}^jP^x_{\alpha\dot{\alpha}}\langle\bar{\Phi}(x)\Phi(y)\rangle\,. \qquad (4.14)$$

The coefficient, ρ, should be independent of the inserted operators so we can fix it by a convenient choice of Φ. We choose $\Phi=\bar{\varphi}_{ij}$ in the abelian theory and we consider

$$\langle\{Q_{k\alpha},[\bar{Q}^l{}_{\dot{\alpha}},\varphi^{ij}(x)]\}\bar{\varphi}_{ij}(y)\rangle=-i\rho\delta_k{}^l\partial^x_{\alpha\dot{\alpha}}\langle\varphi^{ij}(x)\bar{\varphi}_{ij}(y)\rangle. \qquad (4.15)$$

Using the supersymmetry transformations (2.4)

$$[\bar{Q}^k,\varphi^{ij}]=\frac{1}{2}\epsilon^{ijkl}\bar{\lambda}_l\,,\qquad\{Q_j{}^\alpha,\bar{\lambda}_i{}^{\dot{\alpha}}\}=4i\partial^{\dot{\alpha}\alpha}\bar{\varphi}_{ij}\,,$$

we see that $\rho=1$ in agreement with (4.13). We can similarly deduce

$$\langle[\bar{Q}^j{}_{\dot{\alpha}},[Q_{i\alpha},\bar{\Phi}(x)]_\pm]_\mp\Phi(y)\rangle=\delta_i{}^jP^x_{\alpha\dot{\alpha}}\langle\bar{\Phi}(x)\Phi(y)\rangle\,, \qquad (4.17)$$

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directly from the algebra, or by choosing $\Phi = \bar{\varphi}_{ij}$ and using the relations

$$[Q_k, \varphi^{ij}] = \frac{1}{2}(\delta_k{}^j \lambda^i - \delta_k{}^i \lambda^j)\,, \qquad \{\bar{Q}^j{}_{\dot{\alpha}}, \lambda^i{}_\alpha\} = -4i\partial_{\alpha\dot{\alpha}}\varphi^{ij}\,.$$

We now proceed to evaluate the integrals over B_x^ϵ and B_y^ϵ in (4.9). Using the $\mathcal{O}_\tau$ OPE given by (3.49), (3.51), and (3.53), we can reduce the right hand side of (4.9) to a collection of two-point functions, which we now evaluate.

We briefly explain the evaluation of one of the correlators, the remaining ones can be evaluated in a similar way. Consider the a_1 correlator in (3.49),

$$\begin{aligned}
\langle\{Q_i{}^\alpha,[Q_{j\alpha},\{\bar{Q}^i{}_{\dot{\alpha}},[\bar{Q}^{j\dot{\alpha}},\bar{\mathcal{O}}(x)]\}]\}\mathcal{O}(y)\rangle &= -\langle[\bar{Q}^{i\dot{\alpha}},\bar{\mathcal{O}}(x)]\bar{Q}^j{}_{\dot{\alpha}}Q_{j\alpha}[Q_i{}^\alpha,\mathcal{O}(y)]\rangle \\
&= -4i\partial^x_{\alpha\dot{\alpha}}[\bar{Q}^{i\dot{\alpha}},\bar{\mathcal{O}}(x)][Q_i{}^\alpha,\mathcal{O}(y)]\rangle \\
&= 32\partial_x^2\langle\bar{\mathcal{O}}(x)\mathcal{O}(y)\rangle\,, \qquad (4.19)
\end{aligned}$$

where we have twice made use of (4.10). First we chose $\Phi = [Q_i{}^\alpha, \mathcal{O}]$ ($\delta\mathcal{O}$ is a conformal primary by assumption) and then repeated the procedure with $\Phi = \mathcal{O}$. All the remaining correlators can be evaluated in a similar way. We list the results in appendix E.

Noting that derivatives obtained from $\{Q, \bar{Q}\} \sim P_\mu$ also act on the $1/r^2$ and r^μ/r^2 factors in the $\mathcal{O}_\tau$ OPE, we find on substituting (4.19) and (E.1) that

$$\begin{aligned}
\frac{1}{16\pi}\frac{\partial}{\partial\tau}\langle\bar{\mathcal{O}}(x)\mathcal{O}(y)\rangle &= \frac{1}{16\pi}\left\langle\frac{\partial}{\partial\tau}\left(\bar{\mathcal{O}}(x)\mathcal{O}(y)\right)\right\rangle - \frac{2\pi i}{\tau_2}(A+2C)\langle\bar{\mathcal{O}}(x)\mathcal{O}(y)\rangle + \\
&\quad + \frac{i}{4\pi\tau_2}(A+B+C)\partial^2\langle\bar{\mathcal{O}}(x)\mathcal{O}(y)\rangle\left(\int_{B_x^\epsilon}\frac{d^4z}{(z-x)^2} + \int_{B_y^\epsilon}\frac{d^4z}{(z-y)^2}\right) + \\
&\quad + \cdots\,, \qquad (4.20)
\end{aligned}$$

where

$$\begin{aligned}
A &= 2(a_1+a_3) - 3a_2\,, \\
B &= -2(a_4+a_6) + 3a_5 + \frac{9}{2}\left(a_7 - \frac{1}{2}a_8 + a_9\right), \\
C &= i\left(8a_{10} - 16a_{11} + 20a_{12} - 18a_{13} + 24a_{14} + 3a_{15}\right). \qquad (4.21)
\end{aligned}$$

Of the fifteen coefficients $a_1, \ldots, a_{15}$ in the $\mathcal{O}_\tau$ OPE, only the three combinations of (4.21) appear in the final expression. The integrals over B_x^ϵ and B_y^ϵ are each equal to $\pi^2\epsilon^2$; however, their precise values will not be relevant for us. Note that for two-point functions, the coupling dependence is related to the space-time dependence of the same two-point function. This is generically not the case for higher point functions.

We can now make use of the known space-time dependence of low point functions (fixed by conformal invariance) to constrain the unknown coefficients in (4.20). For two non-coincident points x and y and a scalar $\mathcal{O}$, we use the relation

$$\langle\bar{\mathcal{O}}(x)\mathcal{O}(y)\rangle = \frac{\eta(\tau,\bar{\tau})}{|x-y|^{2\Delta(\tau,\bar{\tau})}} \qquad (4.22)$$

where Δ is the (possibly τ-dependent) conformal dimension of $\mathcal{O}$ which we substitute into (4.4). The left hand side of (4.4) is obtained by differentiating (4.22), giving

$$\frac{\partial}{\partial\tau}\langle\bar{\mathcal{O}}(x)\mathcal{O}(y)\rangle = \frac{1}{|x-y|^{2\Delta}}\left\{\frac{\partial\eta}{\partial\tau} - 2\eta\frac{\partial\Delta}{\partial\tau}\ln|x-y|\right\}. \tag{4.23}$$

In order to evaluate the right hand side of (4.4), we use the known form of the correlator of three fields (with $\Delta_x = \Delta_y = \Delta$ and $\Delta_z = 4$), which is again fixed by conformal invariance to be

$$\langle\mathcal{O}_\tau(z)\bar{\mathcal{O}}(x)\mathcal{O}(y)\rangle = \frac{C^\tau(\tau,\bar{\tau})}{|z-x|^4|z-y|^4|x-y|^{2\Delta-4}}\,, \tag{4.24}$$

which is valid for non-coincident points x, y and z. Equating these expressions (including the explicit derivative) gives,

$$\frac{1}{|x-y|^{2\Delta}}\left\{\frac{\partial\eta}{\partial\tau} - 2\eta\frac{\partial\Delta}{\partial\tau}\ln|x-y|\right\} = \frac{i}{4\tau_2}\frac{C^\tau}{|x-y|^{2\Delta-4}}\int d^4z\frac{1}{|z-x|^4|z-y|^4} + \\ + \left\langle\frac{\partial}{\partial\tau}\left(\bar{\mathcal{O}}(x)\mathcal{O}(y)\right)\right\rangle. \tag{4.25}$$

However, this is not quite correct. In (4.24), we have neglected contact terms that arise when $z \to x$ and $z \to y$. These terms play an important role in our following analysis, as we shall discuss shortly.

First we need to evaluate the integral on the right hand side of (4.25). This integral is UV divergent and so needs to be regularized. Following [36], we use differential regularization. Because $z \neq x$ and $z \neq y$ in (4.24) we can use the formula for non-coincident points, giving [36]

$$\int d^4z\frac{1}{|z-x|^4|z-y|^4} = -\frac{\pi^2}{4}\partial^2\left\{\frac{\ln^2|x-y|^2}{|x-y|^2}\right\} = -2\pi^2\frac{1}{|x-y|^4}\left\{1-2\ln|x-y|\right\}. \tag{4.26}$$

The logarithmic term of [36] appears here in the form $\ln(M^2x^2)$, where M is a mass scale. We have set M to one to match the corresponding expression on the left hand side of (4.25). This equation therefore leads to

$$\left(\frac{\partial}{\partial\tau} - \frac{i(\alpha_{\mathcal{O}}+\alpha_{\bar{\mathcal{O}}})}{\tau_2}\right)\eta - 2\eta\frac{\partial\Delta}{\partial\tau}\ln|x-y| = -\frac{i\pi^2C^\tau}{2\tau_2}\left\{1-2\ln|x-y|\right\}, \tag{4.27}$$

so that

$$\left(\frac{\partial}{\partial\tau} - \frac{i(\alpha_{\mathcal{O}}+\alpha_{\bar{\mathcal{O}}})}{\tau_2}\right)\eta = -\frac{i\pi^2}{2\tau_2}C^\tau\,, \qquad \frac{\partial\Delta}{\partial\tau} = -\frac{i\pi^2}{2\tau_2}\frac{C^\tau}{\eta}\,, \tag{4.28}$$

where the coefficients $\alpha_{\mathcal{O}}$ and $\alpha_{\bar{\mathcal{O}}}$ arise from the last term in (4.25). The second relation shows that the conformal dimension depends on the coupling only via the ratio of the correlator of *three* operators to the correlator of *two* operators. In order to see how the contact terms modify (4.25) we now want to compare (4.27) with (4.20).

Using (4.22), we see that (4.20) reduces to

$$\frac{1}{16\pi}\frac{\partial}{\partial\tau}\langle\bar{\mathcal{O}}(x)\mathcal{O}(y)\rangle = \left\{\frac{i(\alpha_{\mathcal{O}}+\alpha_{\bar{\mathcal{O}}})}{16\pi\tau_2} - \frac{2\pi i(A+2C)}{\tau_2}\right\}\frac{\eta}{(x-y)^{2\Delta}} + \tag{4.29}$$

$$+\frac{i(A+B+C)}{\pi\tau_2}\frac{\Delta(\Delta-1)\eta}{(x-y)^{2\Delta+2}}\left(\int_{B_x^\epsilon}\frac{d^4z}{(z-x)^2}+\int_{B_y^\epsilon}\frac{d^4z}{(z-y)^2}\right)+\cdots.$$

We now want to compare (4.25) and (4.29). This is facilitated by treating the z integral in (4.25) in the same manner as in (4.29) by dividing it up into balls, $B_{x_i}^\epsilon$, surrounding each insertion point, x_i, together with the smooth contribution away from the insertion points. The second term on the right hand side of (4.29) arises from a contact term between $\mathcal{O}_\tau$ and the inserted operators. This is precisely the contact term we neglected in (4.25) so we will not be able to match this term. For the other terms, the expansion of (4.25) gives

$$\begin{aligned}\frac{\partial}{\partial\tau}\langle\bar{\mathcal{O}}(x)\mathcal{O}(y)\rangle-\langle\frac{\partial}{\partial\tau}\left(\bar{\mathcal{O}}(x)\mathcal{O}(y)\right) &= \frac{iC^\tau}{4\tau_2(x-y)^{2\Delta-4}}\left[\int_{B_x^\epsilon}\frac{d^4z}{(z-x)^4(z-y)^4}+\right.\\ &\left.+\int_{B_y^\epsilon}\frac{d^4z}{(z-x)^4(z-y)^4}\right]+\cdots\\ &= \frac{iC^\tau}{4\tau_2(x-y)^{2\Delta}}\left[\int_{B_x^\epsilon}\frac{d^4z}{(z-x)^4}+\int_{B_y^\epsilon}\frac{d^4z}{(z-y)^4}\right]+\\ &+\frac{iC^\tau}{4\tau_2(x-y)^{2\Delta+2}}\left[\int_{B_x^\epsilon}\frac{d^4z}{(z-x)^2}+\int_{B_y^\epsilon}\frac{d^4z}{(z-y)^2}\right]+\\ &+O(\frac{1}{(x-y)^{2\Delta+4}})\,.\end{aligned} \tag{4.30}$$

The $O(1/(x-y)^{2\Delta})$ terms contain UV divergent integrals. We can again evaluate the integrals using differential regularization again [36], which gives

$$\int_{B_x^\epsilon}\frac{d^4z}{(z-x)^4}=-\frac{1}{4}\partial_x^2\int_{B_x^\epsilon}d^4z\frac{\ln(z-x)^2M^2}{(z-x)^2}=-\frac{1}{4}\partial_x^2\int_{B_0^\epsilon}d^4z\frac{\ln z^2M^2}{z^2}=0\,, \tag{4.31}$$

since the integral over B_0^ϵ is independent of x. This is in agreement with (4.29). Also terms of $O(1/(x-y)^{2\Delta+4})$ appear with integrals of the form

$$\int_{B_x^\epsilon}d^4z\,,\qquad\int_{B_x^\epsilon}d^4z(z-x)^2\,,\qquad\int_{B_x^\epsilon}d^4z(z-x)^4\,,\qquad\cdots. \tag{4.32}$$

These integrands are regular as $z\to x$, and so they do not appear in the $\mathcal{O}_\tau$ OPE expansion and are absent from (4.29).

Equating the $O(1/(x-y)^{2\Delta+2})$ terms in (4.29) and (4.30) gives

$$\frac{C^\tau}{64\eta}=(A+B+C)\Delta(\Delta-1)\,. \tag{4.33}$$

The contact term contribution in (4.29) together with the explicit τ derivative (accounted for by (4.6)) of the inserted operators modifies the first equation of (4.28)

$$\left(\frac{\partial}{\partial\tau}+\frac{32\pi^2i}{\tau_2}(A+2C)-\frac{i(\alpha_{\mathcal{O}}+\alpha_{\bar{\mathcal{O}}})}{\tau_2}\right)\eta=-\frac{16\pi}{\tau_2}(A+B+C)\Delta(\Delta-1)\eta \tag{4.34}$$

where we have used (4.33). Note that $A+2C$ is generally a coupling dependent function.

The second equation in (4.28) leads to

$$\frac{\partial \Delta}{\partial \tau} = -\frac{16\pi}{\tau_2}(A+B+C)\Delta(\Delta-1)\,. \tag{4.35}$$

We now argue that contact terms cannot modify (4.35). From the left hand side of (4.25), we see that any possible contact term contribution to $\frac{\partial\Delta}{\partial\tau}$ must include a factor of $\ln|x-y|$. However, in finding contact term contributions, we consider integrals over B_x^ϵ (and B_y^ϵ) and pick up the contributions when $z=x$ (and $z=y$). However, these integrals always result in a power series expansion in $1/|x-y|$ and not a $\ln|x-y|$ term. So no contact term is generated.

4.3 Implications for BPS two-point functions

Let us start by considering the case where $\mathcal{O}$ is the superconformal primary of a BPS multiplet; for example, $\mathcal{O}$ can be $\mathcal{O}_p$ for a 1/2 BPS multiplet. Let $\mathcal{O}$ be annihilated by Q_i^α so $[Q_i^\alpha,\mathcal{O}]=0$. In this case, further constraints are implied by superconformal Ward identities. In the supercurrent Ward identity (2.27), take $\Phi^{I_1}=[\bar{Q}^{i\dot\alpha},\mathcal{O}]$ and $\Phi^{I_2}=\mathcal{O}$,[1]

$$\begin{aligned}\partial^\mu\langle J^\alpha_{\mu i}(z)[\bar{Q}^{i\dot\alpha},\mathcal{O}(x)]\mathcal{O}(y)\rangle \;=\; \Big\{&\delta^4(z-x)\langle\{Q_i^\alpha,[\bar{Q}^{i\dot\alpha},\mathcal{O}(x)]\}\mathcal{O}(y)\rangle + \\ &+\delta^4(z-y)\langle[\bar{Q}^{i\dot\alpha},\mathcal{O}(x)][Q_i^\alpha,\mathcal{O}(y)]\rangle\Big\}\,. \end{aligned}\tag{4.36}$$

Using the relation,

$$[Q_i,\mathcal{O}_\tau] = \partial^\mu J_{\mu i} + 2\sigma^\mu\bar\sigma^\nu\partial_\mu J_{\nu i}\,, \tag{4.37}$$

the left hand side of (4.36) reduces to

$$\langle[Q_i{}^\alpha,\mathcal{O}_\tau(z)][\bar{Q}^{i\dot\alpha},\mathcal{O}(x)]\mathcal{O}(y)\rangle - 2(\sigma_\mu\bar\sigma_\nu)_\alpha{}^\beta\partial^{z^\mu}\langle J^\nu_{i\beta}(z)[\bar{Q}^{i\dot\alpha},\mathcal{O}(x)]\mathcal{O}(y)\rangle\,. \tag{4.38}$$

The second term will vanish when we integrate over z by essentially the same argument given around (3.35). The left hand side of (4.36) is then proportional to

$$\partial_x^{\dot\alpha\alpha}\langle\mathcal{O}_\tau(z)\mathcal{O}(x)\mathcal{O}(y)\rangle\,. \tag{4.39}$$

Finally (4.36) yields the coupling independent relation

$$\partial_x^{\dot\alpha\alpha}\langle\mathcal{O}_\tau(z)\mathcal{O}(x)\mathcal{O}(y)\rangle \sim \left[\delta^4(z-x)+\delta^4(z-y)\right]\partial_x^{\dot\alpha\alpha}\langle\mathcal{O}(x)\mathcal{O}(y)\rangle\,. \tag{4.40}$$

Integrating over z gives

$$\int d^4z\langle\mathcal{O}_\tau(z)\mathcal{O}(x)\mathcal{O}(y)\rangle \sim \langle\mathcal{O}(x)\mathcal{O}(y)\rangle \tag{4.41}$$

(we need not worry about removing the ∂_x in going from (4.40) to (4.41) because we know the precise x-dependence of these correlators). So the only contributions from the $\mathcal{O}_\tau$ OPE are contact term contributions which means that

$$A+B+C=0 \qquad \Rightarrow \frac{\partial\Delta}{\partial\tau}=0\,. \tag{4.42}$$

We thus recover the known result that the conformal dimensions of BPS superconformal primary operators are not renormalized.

[1] Note that if $[Q_i^\alpha,\mathcal{O}]=0$ then $[\bar{Q}^{i\dot\alpha},\mathcal{O}]\neq 0$. Otherwise, using $\{Q,\bar{Q}\}\sim\partial_\mu$ gives $\partial_\mu\mathcal{O}=0$ and $\mathcal{O}$ must be proportional to the identity operator.

To fix the metric on this subsector of BPS operators, we need to note that the relation (4.36) is independent of the coupling. This is a rather important point which can be seen as follows. Integrate both sides of (4.36) over z. For the right hand side, the integration is trivial. The left hand side becomes

$$\int d^4z\, \partial^\mu \langle J^\alpha_{\mu i}(z)[\bar{Q}^{i\dot{\alpha}}, \mathcal{O}(x)]\mathcal{O}(y)\rangle = \langle \{Q^\alpha_i, [\bar{Q}^{i\dot{\alpha}}, \mathcal{O}(x)]\}\mathcal{O}(y)\rangle + \langle [\bar{Q}^{i\dot{\alpha}}, \mathcal{O}(x)][Q^\alpha_i, \mathcal{O}(y)]\rangle$$

so the coefficient of proportionality in (4.36) is just a constant. However, we also need to note that the supersymmetry variation of a BPS operator is not quantum corrected, unlike the case of long operators. This follows, essentially, because the divergence of the supercurrent sits in the same anomaly multiplet as T^μ_μ. For an example of quantum corrections to the supersymmetry variation of a long operator, see [20]. The only way a coupling dependence could appear is if x approaches y and a long operator emerges in the OPE of $[\bar{Q}^{i\dot{\alpha}}, \mathcal{O}(x)]$ and $\mathcal{O}(y)$. The quantum corrections to the supervariation of this long operator encodes the coupling dependence of the correlation function. However, in this case, what would remain is the 1-point function of a long operator which vanishes.

The Ward identity has therefore taught us that the only relevant terms in the OPE between $\mathcal{O}_\tau$ and the inserted operators are the tree-level contact terms. So we conclude that,

$$\left(\frac{\partial}{\partial\tau} + \frac{\alpha}{\tau_2}\right)\eta = 0\,, \tag{4.43}$$

for some constant α and that $A + 2C$ is a constant independent of the coupling.

Lastly, we can set α to zero in the following way: we need to examine the solutions to (4.43) and its conjugate equation. A quick inspection reveals that the only solution for η is a single fixed power of τ_2. However, we have the freedom to rescale $\mathcal{O}$ by powers of τ_2 in a way compatible with (4.6) (the value of α_{I_i} shifts by such a rescaling). Using this freedom, we set $\alpha = 0$ in (4.43) and conclude that η is a constant. With this normalization, each BPS superconformal primary has modular weight $(0, 0)$. This is consistent with our expectations from S-duality [4].

The only supermultiplet with a canonical normalization is the current multiplet. In particular, $\mathcal{O}_2$ appears with a single factor of τ_2. In this case, we do not want to use our rescaling freedom to change α. However, using the canonical normalization, we see that the tree-level 2-point function,

$$\langle \mathcal{O}_2(x)\mathcal{O}_2(y)\rangle \sim \frac{1}{|x-y|^4}\,,$$

is coupling independent and non-zero. Therefore, in this case we find $\alpha = 0$ automatically with the canonical normalization. We conclude that 2-point functions of superconformal BPS primary operators are not renormalized. This statement is non-perturbative and applies to 1/2, 1/4 and 1/8 BPS multiplets. We might worry that in an instanton background, half of the supersymmetries are broken so the Ward identities associated with the broken currents are inapplicable. However, we are still free to choose any $J^\alpha_{\mu i}$ in (4.36) which is preserved, and the rest of the argument is unchanged.

We can extend this result to two-point functions of BPS operators which are not superconformal primaries. This follows fairly directly from [4]. First note that we are free to move all the δ and $\bar{\delta}$ operators so that they act on one of the two inserted operators. We then observe that any two-point function of this kind satisfies the following relation

$$\langle \mathcal{O}(x)\delta^n\bar{\delta}^m\mathcal{O}(y)\rangle \sim \delta^{(n,m)}\,\partial_y^n\langle \mathcal{O}(x)\mathcal{O}(y)\rangle\,, \tag{4.44}$$

which follows for the same reasons as (4.10). This two-point function is therefore also not renormalized.

It is worth noting that there can be contact term contributions to two-point correlators. These contact terms are renormalized even for BPS operators, as shown explicitly in [32]. Also, these contact terms can and do appear in correlators like (4.44) even when $n \neq m$ (in which case the correlator vanishes for separated insertion points).

4.4 Implications for BPS three-point functions

Let us consider the three point function $\langle \mathcal{O}^I(x)\mathcal{O}^J(y)\mathcal{O}^K(w)\rangle$ where the operators are BPS superconformal primaries at separated points. Using the result for conformal primaries, we define the ring coefficients $\mathcal{C}^{IJK}$,

$$\langle \mathcal{O}^I(x)\mathcal{O}^J(y)\mathcal{O}^K(w)\rangle = \frac{\mathcal{C}^{IJK}(\tau,\bar{\tau})}{|x-y|^{\Delta_{IJK}}|x-w|^{\Delta_{IKJ}}|y-w|^{\Delta_{JKI}}}\,, \tag{4.45}$$

where $\Delta_{IJK} = \Delta_I + \Delta_J - \Delta_K$ and $\frac{\partial \Delta_I}{\partial \tau} = 0$.

The coupling dependence, which follows from (4.4), is determined by

$$\begin{aligned}\frac{\partial}{\partial\tau}\langle \mathcal{O}^I(x)\mathcal{O}^J(y)\mathcal{O}^K(w)\rangle = &\left\langle \frac{\partial}{\partial\tau}\left(\mathcal{O}^I(x)\mathcal{O}^J(y)\mathcal{O}^K(w)\right)\right\rangle + \\ &+\frac{i}{4\tau_2}\int d^4z\langle \mathcal{O}_\tau(z)\mathcal{O}^I(x)\mathcal{O}^J(y)\mathcal{O}^K(w)\rangle\,.\end{aligned} \tag{4.46}$$

By analogy with the case of the two-point function, consider the Ward identity (2.27) with $\Phi^{I_1} = [\bar{Q}^{i\dot{\alpha}}, \mathcal{O}^I]$, $\Phi^{I_2} = \mathcal{O}^J$ and $\Phi^{I_3} = \mathcal{O}^K$ and where each $\mathcal{O}$ is annihilated by ${Q_i}^\alpha$. Repeating our prior argument gives,

$$\begin{aligned}\partial_x^{\dot{\alpha}\alpha}\langle \mathcal{O}_\tau(z)\mathcal{O}^I(x)\mathcal{O}^J(y)\mathcal{O}^K(w)\rangle \sim &\left[\delta^4(z-x)+\delta^4(z-y)+\delta^4(z-w)\right]\times \\ &\times\partial_x^{\dot{\alpha}\alpha}\langle \mathcal{O}^I(x)\mathcal{O}^J(y)\mathcal{O}^K(w)\rangle\,,\end{aligned} \tag{4.47}$$

with the constant of proportionality again independent of the coupling. Indeed, the constant is just the value of the contact term between $\mathcal{O}_\tau$ and each inserted operator. This same contact term appeared in the determination of BPS two-point functions.

In deriving (4.47), we have moved a Q around the correlation function just as in the discussion around (4.38). If this Q hits a long operator, the relation might be renormalized. Again, a long operator can only emerge from the OPE of two of the inserted BPS operators. However, that would leave a two-point function of a long and a BPS operator which always vanishes. The constant in (4.47) is therefore not renormalized.

As an aside, we should comment that had we considered a correlator of 4 BPS operators, the conclusion would be different. In this case, the Ward identity no longer guarantees the absence of quantum corrections: a long operator can emerge from two short operators but now the resulting three-point function need not vanish. If all possible three-point functions of this kind were to vanish (for other symmetry reasons) then the 4-point function would again be protected from renormalization.

Returning to the three-point function, we note that integrating (4.47) gives,

$$\int d^4z \langle \mathcal{O}_\tau(z) \mathcal{O}^I(x) \mathcal{O}^J(y) \mathcal{O}^K(w) \rangle \sim \langle \mathcal{O}^I(x) \mathcal{O}^J(y) \mathcal{O}^K(w) \rangle \,. \tag{4.48}$$

Thus (4.46) implies that

$$\left(\frac{\partial}{\partial \tau} + \frac{\alpha'}{\tau_2} \right) \mathcal{C}^{IJK} = 0 \tag{4.49}$$

for some constant α'. Normalizing the operators as in section 4.3 so that the two-point functions are independent of the coupling then trivially implies that $\alpha' = 0$; the contribution from the explicit derivative together with the contact term with $\mathcal{O}_\tau$ vanishes separately for each operator. We conclude that the three-point functions of BPS superconformal primaries are not renormalized.

4.4.1 Comments on three-point functions of descendents

It turns out that our non-renormalization proof does not extend simply to 3-point correlators of descendents. For the special case of the current multiplet, the non-renormalization result does extend to descendents. This follows from the analysis of [6] where correlators of descendents were related to $\langle \mathcal{O}_2(x) \mathcal{O}_2(y) \mathcal{O}_2(w) \rangle$ using superconformal symmetry.

For other BPS multiplets, we can demonstrate non-renormalization under the assumption that (anti-)instanton corrections to the 3-point correlator vanish. The argument uses $\mathrm{SL}(2,\mathbb{Z})$ in the following way: any correlator of superconformal descendents is of the form

$$\langle \delta^r \bar{\delta}^{\bar{r}} \mathcal{O}_1(x) \delta^s \bar{\delta}^{\bar{s}} \mathcal{O}_2(y) \delta^t \bar{\delta}^{\bar{t}} \mathcal{O}_3(w) \rangle \,, \tag{4.50}$$

where $\mathcal{O}_1, \mathcal{O}_2$ and $\mathcal{O}_3$ are BPS superconformal primaries. By moving around the δ and $\bar{\delta}$ operators, these correlators can always be put in the form

$$\langle \mathcal{O}_1(x) \delta^m \bar{\delta}^{\bar{n}} \mathcal{O}_2(y) \delta^p \bar{\delta}^{\bar{q}} \mathcal{O}_3(w) \rangle \,, \tag{4.51}$$

or its space-time derivatives. So it is good enough to analyze the coupling dependence of (4.51). Note that neither $\delta^m \bar{\delta}^{\bar{n}}$ nor $\delta^p \bar{\delta}^{\bar{q}}$ yield conformal descendents by definition (if they do, they would merely become space-time derivatives of correlators like (4.51)). To study the coupling dependence of (4.51), we consider the OPE of $\mathcal{O}_1(x)$ and $\delta^m \bar{\delta}^{\bar{n}} \mathcal{O}_2(y)$ as $x \to y$. Assuming (4.51) is non-vanishing (otherwise there is nothing to prove), we see that the OPE is given by

$$\mathcal{O}_1(x) \delta^m \bar{\delta}^{\bar{n}} \mathcal{O}_2(y) \sim \sum_k g_k(x-y) f_k(g^2_{\mathrm{YM}}, \partial_\mu) \delta^{u_k} \bar{\delta}^{v_k} \mathcal{O}_3(y) + \cdots \,, \tag{4.52}$$

where the omitted terms involve long operators as well as BPS operators which are not (super)conformal descendents of $\mathcal{O}_3$. We assume no instanton corrections so that f_k is independent of θ. All the (super)conformal descendents of $\mathcal{O}_3$ are included in the sum. In the k-th term, u_k and v_k are non-negative integers and the OPE coefficients $f_k(g^2_{\rm YM}, \partial_\mu)$ are a priori functions of the coupling. Clearly, the omitted BPS and long operators do not contribute to (4.51). In fact, substituting (4.52) into (4.51) gives (as $x \to y$)

$$\langle \mathcal{O}_1(x)\delta^m\bar{\delta}^{\bar{n}}\mathcal{O}_2(y)\delta^p\bar{\delta}^{\bar{q}}\mathcal{O}_3(w)\rangle = \sum_k \delta^{(u_k+p,v_k+q)} g_k(x-y) f_k(g^2_{\rm YM},\partial_\mu)\partial_y^{u_k+p}\langle \mathcal{O}_3(y)\mathcal{O}_3(w)\rangle\,. \tag{4.53}$$

Since $\langle \mathcal{O}_3(y)\mathcal{O}_3(w)\rangle$ is coupling independent, in order to demonstrate the coupling independence of (4.51), we need to show that $f_k(g^2_{\rm YM},\partial_\mu)$ is independent of coupling for every k.

To see this, we consider the $\mathrm{SL}(2,\mathbb{Z})$ transformation properties of (4.52). The BPS superconformal primaries $\mathcal{O}_1, \mathcal{O}_2$ and $\mathcal{O}_3$, are $\mathrm{SL}(2,\mathbb{Z})$ invariant. Also both sides of (4.52) transform covariantly under $\mathrm{SL}(2,\mathbb{Z})$ because the left hand side is constructed purely out of BPS operators (Every omitted term involving a long operator is also expected to transform covariantly, though the coupling dependent OPE coefficient and the operator need not do so individually). Let δ and $\bar{\delta}$ transform as modular forms of weights $(\mu,-\mu)$ and $(-\mu,\mu)$ respectively. That they have weights of the form $(\nu,-\nu)$ follows because $\delta\bar{\delta} \sim \partial_\mu$ as far as the $\mathrm{SL}(2,\mathbb{Z})$ structure is concerned and ∂_μ is $\mathrm{SL}(2,\mathbb{Z})$ invariant. Here the specific value of μ (which is 1/4) is not needed. Now $f_k(g^2_{\rm YM},\partial_\mu)$ is also a modular form but, by assumption, it is independent of $\tau_1 = \theta/2\pi$. Hence, it can only be a power of τ_2. So, let $f_k(g^2_{\rm YM},\partial_\mu) = \tau_2^k f_k(\partial_\mu)$. So the left hand side of (4.52) transforms as a modular form of weights $((m-n)\mu, -(m-n)\mu)$, while the k-th term on the right hand side of (4.52) transforms with weights $(-k+(u_k-v_k)\mu, -k-(u_k-v_k)\mu)$. Equating these expressions, we find that

$$(m-n)\mu = -k+(u_k-v_k)\mu\,, \qquad (m-n)\mu = k+(u_k-v_k)\mu\,, \tag{4.54}$$

which leads to $k=0$. Hence, (4.51) is independent of the coupling and (4.51) must have an equal number of δ and $\bar{\delta}$ insertions. Thus the 3-point function of non-coincident BPS superconformal descendents is not renormalized if f_k in (4.52) is independent of τ_1.

4.4.2 A simplified integration formula

These results lead to a pretty formula for the integrated OPE of $\mathcal{O}_\tau$ with a BPS operator $\mathcal{O}$. Consider the relation,

$$\frac{\partial}{\partial\tau}\langle \mathcal{O}(x)\mathcal{O}'_i(y)\rangle = \left\langle \frac{\partial}{\partial\tau}\left\{\mathcal{O}(x)\mathcal{O}'_i(y)\right\}\right\rangle + \frac{i}{4\tau_2}\int d^4z\langle \mathcal{O}_\tau(z)\mathcal{O}(x)\mathcal{O}'_i(y)\rangle\,, \tag{4.55}$$

where $\mathcal{O}'_i(y)$ is any BPS operator. The left hand side vanishes by our prior non-renormalization argument. The first term of the right hand side is zero if $\mathcal{O}'_i(y)$ is not in the same supermultiplet as $\mathcal{O}(x)$. In this case, the integrated 3-point function vanishes.

If $\mathcal{O}'_i(y)$ is in the same supermultiplet as $\mathcal{O}(x)$ then the explicit derivative term can be non-vanishing but is always proportional to $1/\tau_2$. This term must be cancelled by the integrated 3-point function which must be exact at tree-level. It is easy to check that $\int d^4z\mathcal{O}_\tau(z)$ is zero on-shell so the 3-point function is purely a contact term.[2] The BPS operators in the integrated OPE between $\mathcal{O}_\tau$ and $\mathcal{O}$ are therefore in the same supermultiplet as $\mathcal{O}$ and arise only from tree-level contact terms in the OPE (up to integration by parts). We therefore conclude that,

$$\int d^4z\mathcal{O}_\tau(z)\mathcal{O}(x) \sim \sum_i \mathcal{O}'_i(x) + \cdots\,, \tag{4.56}$$

where the omitted terms involve long and semi-short operators.

4.5 A comment about generic three-point functions

The coupling dependence of generic correlators is given by

$$\begin{aligned}
\frac{\partial}{\partial\tau}\langle\mathcal{O}(x)\widehat{\mathcal{O}}(y)\widetilde{\mathcal{O}}(w)\rangle &= \langle\frac{\partial}{\partial\tau}\Big(\mathcal{O}(x)\widehat{\mathcal{O}}(y)\widetilde{\mathcal{O}}(w)\Big)\rangle + \frac{i}{4\tau_2}\int d^4z\langle\mathcal{O}_\tau(z)\mathcal{O}(x)\widehat{\mathcal{O}}(y)\widetilde{\mathcal{O}}(w)\rangle \\
&= \langle\frac{\partial}{\partial\tau}\Big(\mathcal{O}(x)\widehat{\mathcal{O}}(y)\widetilde{\mathcal{O}}(w)\Big)\rangle \\
&\quad + \frac{i}{4\tau_2}\int_{B_x^\epsilon} d^4z\langle\mathcal{O}_\tau(z)\mathcal{O}(x)\widehat{\mathcal{O}}(y)\widetilde{\mathcal{O}}(w)\rangle + \cdots\,,
\end{aligned} \tag{4.57}$$

where $\cdots$ includes integrals over B_y^ϵ and B_w^ϵ, and over the region outside the balls. Again, because we know the precise form of the space-time dependence of three-point functions, it might be possible to learn about these correlators from the ball approximation.

After using the $\mathcal{O}_\tau$ OPE, we note that the correlators on the right hand side generically involve different operators from those appearing on the left hand side of (4.57). In order to see this, it is enough to consider a particular term in the integral over B_x^ϵ. Consider the a_2 term in (3.49) which gives a contribution

$$\frac{ia_2}{4\tau_2}\int_{B_x^\epsilon}\frac{d^4z}{(z-x)^2}\langle\{Q_i{}^\alpha,[\bar{Q}^i{}_{\dot\alpha},\{\bar{Q}^{j\dot\alpha},[Q_{j\alpha},\mathcal{O}(x)]\}]\}\widehat{\mathcal{O}}(y)\widetilde{\mathcal{O}}(w)\rangle\,. \tag{4.58}$$

For example, take $\mathcal{O}=\mathcal{O}_2$ and take $\widehat{\mathcal{O}}$ and $\widetilde{\mathcal{O}}$ to be superconformal primaries of long multiplets. Then from (2.5), very schematically, we see that (4.58) contributes

$$\frac{a_2}{\tau_2}\int_{B_x^\epsilon}\frac{1}{(z-x)^2}\Big[\langle T_{\mu\nu}(x)\widehat{\mathcal{O}}(y)\widetilde{\mathcal{O}}(w)\rangle + \langle\partial_\mu R_\nu\widehat{\mathcal{O}}(y)\widetilde{\mathcal{O}}(w)\rangle + \langle\partial_\mu\partial_\nu\mathcal{O}_2(x)\widehat{\mathcal{O}}(y)\widetilde{\mathcal{O}}(w)\rangle\Big] + \cdots\,,$$

where $\cdots$ includes terms that involve derivatives acting on $\frac{1}{(z-x)^2}$. So we see that the operators involved differ from those in $\langle\mathcal{O}(x)\widehat{\mathcal{O}}(y)\widetilde{\mathcal{O}}(w)\rangle$, but they are all operators in the same supermultiplet (here $\mathcal{O}_2$) as the original one. Clearly, this is also true for higher point functions.

[2] There is a caveat worth mentioning: to see that $\int d^4z\mathcal{O}_\tau(z)$ is zero on-shell requires integrating by parts and throwing away a boundary term. In euclidean space, we could consider an instanton background in which we might need to consider this boundary term.

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A. The superconformal algebra

In this appendix, we briefly review the superconformal algebra in four dimensions. With metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$, the conformal algebra SO(4,2) in $d = 4$ is given by the commutation relations

$$\begin{gathered}
[M_{\mu\nu}, P_\sigma] = i(\eta_{\mu\sigma}P_\nu - \eta_{\nu\sigma}P_\mu)\,, \qquad [M_{\mu\nu}, K_\sigma] = i(\eta_{\mu\sigma}K_\nu - \eta_{\nu\sigma}K_\mu)\,, \\
[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho})\,, \\
[D, P_\mu] = iP_\mu, \qquad [D, K_\mu] = -iK_\mu\,, \qquad [K_\mu, P_\nu] = -2iM_{\mu\nu} - 2i\eta_{\mu\nu}D\,,
\end{gathered} \tag{A.1}$$

where $M_{\mu\nu}, P_\mu, K_\mu$ and D are the generators of Lorentz transformations, translations, special conformal transformations and dilations, respectively. In $d = 4$, the conformal algebra SO(4,2) can be extended to the superconformal algebra SU(2,2|4) by the inclusion of the supersymmetry charges $Q_{i\alpha}, \bar{Q}^i_{\dot\alpha}$ and superconformal charges $S^{i\alpha}, \bar{S}^{\dot\alpha}_i$, where $i = 1,\ldots,4$. These charges transform in the (anti-)fundamental representation of the global SU(4) R-symmetry group. We denote the generators of the R-symmetry group by $R^i{}_j$ subject to the condition $R^i{}_i = 0$. The non-zero anti-commutation relations between the supersymmetry charges and the superconformal charges are given by

$$\begin{gathered}
\{Q_{i\alpha}, \bar{Q}^j_{\dot\alpha}\} = 2\delta^j_i \sigma^\mu_{\alpha\dot\alpha}P_\mu\,, \qquad \{\bar{S}^{\dot\alpha}_i, S^{j\alpha}\} = 2\delta^j_i \bar\sigma^{\dot\alpha\alpha}_\mu K^\mu\,, \\
\{Q_{i\alpha}, S^{j\beta}\} = 4\left[\delta^j_i\left(M_\alpha{}^\beta - \frac{i}{2}\delta_\alpha{}^\beta D\right) - \delta_\alpha{}^\beta R^j_i\right], \\
\{\bar{S}^{\dot\alpha}_i, \bar{Q}^j_{\dot\beta}\} = 4\left[\delta^j_i\left(\bar{M}^{\dot\alpha}{}_{\dot\beta} + \frac{i}{2}\delta^{\dot\alpha}{}_{\dot\beta}D\right) - \delta^{\dot\alpha}{}_{\dot\beta}R^j_i\right],
\end{gathered} \tag{A.2}$$

where the Lorentz generators are expressed in a spinorial basis,

$$M_\alpha{}^\beta = -\frac{i}{4}(\sigma^\mu\bar\sigma^\nu)_\alpha{}^\beta M_{\mu\nu}\,, \qquad \bar{M}^{\dot\alpha}{}_{\dot\beta} = -\frac{i}{4}(\bar\sigma^\mu\sigma^\nu)^{\dot\alpha}{}_{\dot\beta}M_{\mu\nu}\,.$$

The non-zero commutation relations involving $M_\alpha{}^\beta$, $\bar{M}^{\dot\alpha}{}_{\dot\beta}$, the supersymmetry charges, and superconformal charges are given by

$$\begin{aligned}
[M_\alpha{}^\beta, Q_{i\gamma}] &= \delta_\gamma{}^\beta Q_{i\alpha} - \frac{1}{2}\delta_\alpha{}^\beta Q_{i\gamma}, \qquad & [M_\alpha{}^\beta, S^{i\gamma}] &= -\delta_\alpha{}^\gamma S^{i\beta} + \frac{1}{2}\delta_\alpha{}^\beta S^{i\gamma}\,, \\
[\bar{M}^{\dot\alpha}{}_{\dot\beta}, \bar{Q}^i_{\dot\gamma}] &= -\delta^{\dot\alpha}{}_{\dot\gamma}\bar{Q}^i_{\dot\beta} + \frac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta}\bar{Q}^i_{\dot\gamma}\,, \qquad & [\bar{M}^{\dot\alpha}{}_{\dot\beta}, \bar{S}^{\dot\gamma}_i] &= \delta^{\dot\gamma}{}_{\dot\beta}\bar{S}^{\dot\alpha}_i - \frac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta}\bar{S}^{\dot\gamma}_i\,,
\end{aligned} \tag{A.3}$$

while $[M_{\mu\nu}, M_{\rho\sigma}]$ in the spinorial basis gives

$$[M_\alpha{}^\beta, M_\gamma{}^\delta] = \delta_\gamma{}^\beta M_\alpha{}^\delta - \delta_\alpha{}^\delta M_\gamma{}^\beta\,,$$

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$$[\bar{M}^{\dot\alpha}{}_{\dot\beta}, \bar{M}^{\dot\gamma}{}_{\dot\delta}] = -\delta^{\dot\alpha}{}_{\dot\delta}\bar{M}^{\dot\gamma}{}_{\dot\beta} + \delta^{\dot\gamma}{}_{\dot\beta}\bar{M}^{\dot\alpha}{}_{\dot\delta}\,. \tag{A.4}$$

The non-zero commutation relations involving the dilation operator, the supersymmetry charges and the superconformal charges are given by:

$$\begin{aligned} [D, Q_{i\alpha}] &= \frac{i}{2}Q_{i\alpha}\,, \qquad & [D, \bar{Q}^i_{\dot\alpha}] &= \frac{i}{2}\bar{Q}^i_{\dot\alpha}\,, \\ [D, S^{i\alpha}] &= -\frac{i}{2}S^{i\alpha}\,, \qquad & [D, \bar{S}^{\dot\alpha}_i] &= -\frac{i}{2}\bar{S}^{\dot\alpha}_i\,. \end{aligned} \tag{A.5}$$

Those involving P_μ and K_μ are

$$\begin{aligned} [K^\mu, Q_{i\alpha}] &= -\sigma^\mu_{\alpha\dot\alpha}\bar{S}^{\dot\alpha}_i\,, \qquad & [K^\mu, \bar{Q}^i_{\dot\alpha}] &= S^{i\alpha}\sigma^\mu_{\alpha\dot\alpha}\,, \\ [P_\mu, \bar{S}^{\dot\alpha}_i] &= -\bar\sigma^{\dot\alpha\alpha}_\mu Q_{i\alpha}\,, \qquad & [P_\mu, S^{i\alpha}] &= \bar{Q}^i_{\dot\alpha}\bar\sigma^{\dot\alpha\alpha}_\mu\,. \end{aligned} \tag{A.6}$$

Finally, we list the defining relations for the R-symmetry generators

$$[R^i{}_j, R^k{}_l] = \delta^k{}_j R^i{}_l - \delta^i{}_l R^k{}_j\,, \tag{A.7}$$

and their commutation relations with the super(conformal) charges

$$\begin{aligned} [R^i{}_j, Q_{k\alpha}] &= \delta^i_k Q_{j\alpha} - \frac{1}{4}\delta^i_j Q_{k\alpha}\,, \qquad & [R^i{}_j, \bar{Q}^k_{\dot\alpha}] &= -\delta^k_j \bar{Q}^i_{\dot\alpha} + \frac{1}{4}\delta^i_j \bar{Q}^k_{\dot\alpha}\,, \\ [R^i{}_j, S^{k\alpha}] &= -\delta^k_j S^{i\alpha} + \frac{1}{4}\delta^i_j S^{k\alpha}\,, \qquad & [R^i{}_j, \bar{S}^{\dot\alpha}_k] &= \delta^i_k \bar{S}^{\dot\alpha}_j - \frac{1}{4}\delta^i_j \bar{S}^{\dot\alpha}_k\,. \end{aligned} \tag{A.8}$$

For unitary representations of the superconformal algebra, these operators satisfy the conditions

$$Q_{i\alpha}{}^\dagger = \bar{Q}^i_{\dot\alpha}\,, \qquad S^{i\alpha\,\dagger} = \bar{S}^{\dot\alpha}_i\,, \qquad M_\alpha{}^{\beta\,\dagger} = \bar{M}^{\dot\beta}{}_{\dot\alpha}\,, \qquad R^i_j{}^\dagger = R^j{}_i\,. \tag{A.9}$$

B. The properties and construction of short multiplets

The short multiplet has special properties which can be deduced from the superconformal algebra. For example, states have conformal dimensions that are protected from renormalization. In this appendix, we will deduce some of these properties while describing the explicit construction of the multiplet. Our discussion again follows [22]. We will need these explicit results in the main text so, for completeness, we list them here.

First, we give explicit representatives for $M_\alpha{}^\beta, \bar{M}^{\dot\alpha}{}_{\dot\beta}$ and $R^i{}_j$. Writing

$$[M_\alpha{}^\beta] = \begin{pmatrix} J_3 & J_+ \\ J_- & -J_3 \end{pmatrix}, \qquad [\bar{M}^{\dot\alpha}{}_{\dot\beta}] = \begin{pmatrix} \bar{J}_3 & \bar{J}_+ \\ \bar{J}_- & -\bar{J}_3 \end{pmatrix}, \tag{B.1}$$

where $J_3, J_\pm$ (and $\bar{J}_3, \bar{J}_\pm$) satisfy the standard commutation relations of SU(2), we see that (B.1) satisfies the commutation relations (A.4).

We can also express R^i_j in terms of the $\mathrm{SU}(4)_R$ generators in the Chevalley basis. That is, for every simple root α^i $(i = 1, 2, 3)$, we take ladder operators $E^{\pm i} = E^{\pm\alpha^i}$ and a Cartan generator h^i chosen to satisfy

$$[h^i, h^j] = 0\,, \qquad [E^{+i}, E^{-i}] = \delta_{ij}h^j\,, \qquad [h_i, E^{\pm j}] = \pm A_{ij}E^{\pm j}\,, \tag{B.2}$$

where

$$A_{ij} = 2\frac{(\alpha^i, \alpha^j)}{|\alpha^j|^2}\,, \tag{B.3}$$

are elements of the Cartan matrix of SU(4) given by

$$[A_{ij}] = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{B.4}$$

We can express any weight vector $\vec{\lambda}$ in terms of the fundamental weights with integral coefficients specified by the Dynkin label $[\lambda_1, \lambda_2, \lambda_3]$. Acting on the weight vector, we note that

$$h^i|\vec{\lambda}\rangle = \lambda_i|\vec{\lambda}\rangle\,. \tag{B.5}$$

Thus every representation has a highest weight state satisfying

$$h^i|\lambda_1, \lambda_2, \lambda_3\rangle^{hw} = \lambda_i|\lambda_1, \lambda_2, \lambda_3\rangle^{hw}\,, \tag{B.6}$$

and

$$E^{+i}|\lambda_1, \lambda_2, \lambda_3\rangle^{hw} = 0\,. \tag{B.7}$$

We now construct the matrices $R^i{}_j$

$$[R^i{}_j] = \begin{pmatrix} \frac{1}{4}(3h_1 + 2h_2 + h_3) & E_1^+ & [E_1^+, E_2^+] & [E_1^+, [E_2^+, E_3^+]] \\ E_1^- & \frac{1}{4}(-h_1 + 2h_2 + h_3) & E_2^+ & [E_2^+, E_3^+] \\ -[E_1^-, E_2^-] & E_2^- & -\frac{1}{4}(h_1 + 2h_2 - h_3) & E_3^+ \\ [E_1^-, [E_2^-, E_3^-]] & -[E_2^-, E_3^-] & E_3^- & -\frac{1}{4}(h_1 + 2h_2 + 3h_3) \end{pmatrix}. \tag{B.8}$$

It is not hard to check that the defining relations (A.7) are satisfied by these representatives.

Now that we have explicit forms for the generators, we see that the the last two equations of (A.2) yield non-trivial constraints on the conformal dimension of a short superconformal primary state. Using the defining property (2.1), we see that for $i = 1, 2$

$$\begin{aligned} \frac{1}{4}\{Q_{i\alpha}, S^{l\beta}\}|k,p,q;j,\bar{\jmath}\rangle^{\mathrm{hw}} &= \begin{pmatrix} \frac{\Delta}{2}\delta^l_i - R^l{}_i + j\delta^l_i & 0 \\ 0 & \frac{\Delta}{2}\delta^l_i - R^l{}_i - j\delta^l_i \end{pmatrix} |k,p,q;j,\bar{\jmath}\rangle^{\mathrm{hw}} + \\ &\quad + \delta^l_i \begin{pmatrix} 0 & 0 \\ \sqrt{2j} & 0 \end{pmatrix} |k,p,q;j-1,\bar{\jmath}\rangle^{\mathrm{hw}}, \\ &= 0\,. \end{aligned} \tag{B.9}$$

The conformal dimension, Δ, satisfies

$$D|k,p,q;j,\bar{\jmath}\rangle^{\mathrm{hw}} = i\Delta|k,p,q;j,\bar{\jmath}\rangle^{\mathrm{hw}}\,.$$

We see that $j = 0$ and that

$$\left(\frac{\Delta}{2}\delta^l_i - R^l{}_i\right)|k,p,q;0,\bar{\jmath}\rangle^{\mathrm{hw}} = 0 \tag{B.10}$$

leading to

$$\Delta = \frac{1}{2}(2p+q)\,, \qquad k=0\,. \tag{B.11}$$

Similarly, from the $\bar{Q}^j_{\dot\alpha}$ constraint for $j=3,4$, we find that $\bar{\jmath}=0$, $q=0$ and $\Delta=p$. So the superconformal primary state of a short multiplet is given by $[0,p,0]_{(0,0)}$ with conformal dimension $\Delta=p$.

We now summarize the construction of the short multiplets by acting with the superymmetry charges $Q_{i\alpha}$ and $\bar{Q}^j_{\dot\alpha}$ ($i=3,4$; $j=1,2$) on the superconformal primary state $[0,p,0]_{(0,0)}$. It will be very convenient to use the Racah-Speiser algorithm for decomposing tensor products of representations (see appendix B of [22] for a review and various applications). The statement of the algorithm goes as follows: given two representations $R_{\underline{\Lambda}}$ and $R_{\underline{\Lambda}'}$ (where $R_{\underline{\Lambda}}$ is the representation with highest weight vector $\underline{\Lambda}=[\lambda_1,\cdots,\lambda_r]$, with r the rank of the group), the tensor product is given by

$$R_{\underline{\Lambda}}\otimes R_{\underline{\Lambda}'} \simeq \sum_{\lambda\in V_{\underline{\Lambda}'}} R_{\underline{\Lambda}+\underline{\lambda}}\,, \tag{B.12}$$

where $V_{\underline{\Lambda}'}$ consists of all the weight vectors for all the states in $R_{\underline{\Lambda}'}$.

The algorithm further tells us that on the right hand side of (B.12),

$$R_{\underline{\lambda}} = \mathrm{sign}(\sigma) R_{\underline{\lambda}^\sigma}\,, \qquad \lambda^\sigma = \sigma(\lambda+\rho)-\rho\,. \tag{B.13}$$

Here σ is an element of the Weyl group and ρ is the Weyl vector. So using this algorithm we can construct the tensor product of two representations, where from (B.13), we note that representations for which $\lambda=\lambda^\sigma$ and $\mathrm{sign}(\sigma)=-1$ vanish from the right hand side of (B.12) and so do not exist in the tensor product decomposition.

In our applications, it tells us that when we are acting with the various supercharges on a general representation $[k,p,q]_{(j,\bar{\jmath})}$, we naively get all possible representations $[k',p',q']_{(j',\bar{\jmath}')}$ obtained by adding the weights of the supercharges to $[k,p,q]_{(j,\bar{\jmath})}$. Of the resulting representations, the ones where $(k',p',q',j',\bar{\jmath}')$ are all non-negative are to be kept. The other representations will have negative Dynkin labels. Some them vanish identically using (B.13). The others we can remove using the equations of motion or conservation laws (we will see an example of this shortly).

So we need to know the Dynkin labels of the various supercharges. They can be obtained by computing $[h^i,Q_{j\alpha}]$, $[h^i,S^{j\alpha}]$ which gives,

$$\begin{aligned}
Q_{1\alpha} &\sim [1,0,0]_{(\pm\frac{1}{2},0)}\,, & Q_{2\alpha} &\sim [-1,1,0]_{(\pm\frac{1}{2},0)}\,,\\
Q_{3\alpha} &\sim [0,-1,1]_{(\pm\frac{1}{2},0)}\,, & Q_{4\alpha} &\sim [0,0,-1]_{(\pm\frac{1}{2},0)}\,,\\
S^{1\alpha} &\sim [-1,0,0]_{(\pm\frac{1}{2},0)}\,, & S^{2\alpha} &\sim [1,-1,0]_{(\pm\frac{1}{2},0)}\,,\\
S^{3\alpha} &\sim [0,1,-1]_{(\pm\frac{1}{2},0)}\,, & S^{4\alpha} &\sim [0,0,1]_{(\pm\frac{1}{2},0)}\,,
\end{aligned} \tag{B.14}$$

and for the conjugates,

$$\begin{aligned}
\bar{Q}^1_{\dot\alpha} &\sim [-1,0,0]_{(0,\pm\frac{1}{2})}\,, & \bar{Q}^2_{\dot\alpha} &\sim [1,-1,0]_{(0,\pm\frac{1}{2})}\,,\\
\bar{Q}^3_{\dot\alpha} &\sim [0,1,-1]_{(0,\pm\frac{1}{2})}, & \bar{Q}^4_{\dot\alpha} &\sim [0,0,1]_{(0,\pm\frac{1}{2})}\,,\\
\bar{S}^{\dot\alpha}_1 &\sim [1,0,0]_{(0,\pm\frac{1}{2})}, & \bar{S}^{\dot\alpha}_2 &\sim [-1,1,0]_{(0,\pm\frac{1}{2})}\,,\\
\bar{S}^{\dot\alpha}_3 &\sim [0,-1,1]_{(0,\pm\frac{1}{2})}, & \bar{S}^{\dot\alpha}_4 &\sim [0,0,-1]_{(0,\pm\frac{1}{2})}\,.
\end{aligned} \tag{B.15}$$

We can now go ahead with the construction of the short multiplet. Acting on the superconformal primary state with the Q operators yields

$$[0,p,0]_{(0,0)} \overset{Q}{\to} [0,p-1,1]_{(\frac{1}{2},0)} \overset{Q^2}{\to} \begin{matrix}[0,p-2,2]_{(0,0)}\\ [0,p-1,1]_{(0,0)}\end{matrix} \overset{Q^3}{\to} [0,p-2,1]_{(\frac{1}{2},0)} \overset{Q^4}{\to} [0,p-2,0]_{(0,0)}\,, \tag{B.16}$$

while acting with the $\bar{Q}$ operators on the highest weight states of the various representations yields

$$[0,p,q]_{(j,0)} \overset{\bar{Q}}{\to} [1,p-1,q]_{(j,\frac{1}{2})} \overset{\bar{Q}^2}{\to} \begin{matrix}[2,p-2,q]_{(j,0)}\\ [0,p-1,q]_{(j,1)}\end{matrix} \overset{\bar{Q}^3}{\to} [1,p-2,q]_{(j,\frac{1}{2})} \overset{\bar{Q}^4}{\to} [0,p-2,q]_{(j,0)}\,. \tag{B.17}$$

These results lead to diagram (2.2) displayed in the main text.

C. The subleading terms in the $\delta^2\mathcal{O}_2$ OPE

In this appendix, we determine the subleading singular terms in (3.39). One way to obtain all the remaining terms in (3.39) would be to consider all the less singular terms in (3.2) and (3.16) and repeat our prior analysis. However, that is a rather complicated route since there are many subleading terms in (3.2) and (3.16)!

So we shall proceed by a different route. Because in (3.39), the leading term goes like $\sim 1/|z-x|^2$, all the possible subleading terms go like $\sim 1/|z-x|$. So the number of these terms is far less than the number of subleading terms in (3.2) or (3.16). Also, as discussed before, we saw from (3.28) that only terms with specific properties under $SO(3,1)$ and $SU(4)_R$ arise in the OPE — this drastically reduces the possible subleading terms in (3.39). So we shall directly write down all possible terms that go like $\sim 1/|z-x|$ in (3.39) consistent with the various symmetries. This will give the full OPE. The leading terms in (3.39) can be written schematically as,

$$\frac{1}{z^2}\delta^2\Phi \qquad \text{and} \qquad \frac{1}{z^2}\bar{\delta}^2\Phi\,. \tag{C.1}$$

At $O(1/(z-x))$, the following terms are the only possibilities consistent with the $SO(3,1)$ and $SU(4)_R$ structure of (3.39):

$$\frac{1}{z}\bar{\delta}^2\partial_\mu\Phi\,, \qquad \frac{1}{z}\delta^2\partial_\mu\Phi\,, \qquad \frac{1}{z}\bar{\delta}^3\delta\Phi \qquad \text{and} \qquad \frac{1}{z}\delta^3\bar{\delta}\Phi\,. \tag{C.2}$$

We now list the $O(1/(z-x))$ terms in the OPE for each of these cases (we use r^μ to denote $(z-x)^\mu$).

(i) $\frac{1}{z}\,\bar\delta^2\partial_\mu\Phi$**:**

$$\begin{aligned}
\Sigma_\alpha{}^{\beta ij}(z)\Phi^I(x) \sim\ & \delta_\alpha{}^\beta\frac{r^\mu}{r^2}\partial_\mu[\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\alpha},\Phi^I(x)]_\pm]_\mp \\
&+(\sigma_{\mu\nu})_\alpha{}^\beta(\bar\sigma^{\mu\rho})^{\dot\alpha}{}_{\dot\beta}\frac{r^\nu}{r^2}\partial_\rho[\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\beta},\Phi^I(x)]_\pm]_\mp \\
&+i\delta_\alpha{}^\beta(S^{\mu\nu})^I{}_J\frac{r_\mu}{r^2}\partial_\nu[\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\alpha},\Phi^J(x)]_\pm]_\mp \\
&+i\delta_\alpha{}^\beta\epsilon_{\mu\nu\rho\lambda}(S^{\mu\nu})^I{}_J\frac{r^\rho}{r^2}\partial^\lambda[\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\alpha},\Phi^J(x)]_\pm]_\mp \\
&+i(\sigma_{\mu\nu})_\alpha{}^\beta(S^{\mu\lambda})^I{}_J(\bar\sigma_{\rho\lambda})^{\dot\alpha}{}_{\dot\beta}\frac{r^\nu}{r^2}\partial^\rho[\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\beta},\Phi^J(x)]_\pm]_\mp \\
&+i\epsilon^{\mu\nu\rho\sigma}(\sigma_{\mu\lambda})_\alpha{}^\beta(S_{\nu\omega})^I{}_J(\bar\sigma_{\rho\tau})^{\dot\alpha}{}_{\dot\beta}\frac{r^\lambda r^\tau r^\omega}{r^4}\partial_\sigma[\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\beta},\Phi^J(x)]_\pm]_\mp \qquad \text{(C.3)}\\
&+i(\sigma_{\mu\nu})_\alpha{}^\beta(S^{\mu\lambda})^I{}_J(\bar\sigma_{\rho\tau})^{\dot\alpha}{}_{\dot\beta}\frac{r^\nu r^\rho r_\lambda}{r^4}\partial^\tau[\bar Q^i_{\dot\alpha},[\bar Q^{j\dot\beta},\Phi^J(x)]_\pm]_\mp+\text{permutations}\,.
\end{aligned}$$

Unlike leading order where there are no $\mathcal{E}$ type $S_{\mu\nu}$ terms, there are $S_{\mu\nu}$ terms of both $\mathcal{E}$ and B type at this order. There are several B type $S_{\mu\nu}$ terms — for brevity, we have written one for each distinct structure, with the remaining independent terms generated by permutations. For example, by permutation, we should include terms like

$$i(\sigma_{\mu\nu})_\alpha{}^\beta(S^{\mu\lambda})^I{}_J(\bar\sigma^{\nu\rho})^{\dot\alpha}{}_{\dot\beta}\frac{r_\lambda}{r^2}\partial_\rho[\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\beta},\Phi^J(x)]_\pm]_\mp\,, \qquad \text{(C.4)}$$

which is a permutation of the fifth term of (C.3). Also among the permutations is

$$i\epsilon^{\mu\nu\rho\sigma}(\sigma_{\mu\lambda})_\alpha{}^\beta(S_{\nu\omega})^I{}_J(\bar\sigma_{\rho\tau})^{\dot\alpha}{}_{\dot\beta}\frac{r^\lambda r^\tau r_\sigma}{r^4}\partial^\omega[\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\beta},\Phi^J(x)]_\pm]_\mp \qquad \text{(C.5)}$$

which is related to the sixth term of (C.3).

(ii) $\frac{1}{z}\,\delta^2\partial_\mu\Phi$**:**

$$\begin{aligned}
\Sigma_\alpha{}^{\beta ij}(z)\Phi^I(x) \sim\ & \epsilon^{ijkl}\frac{r^\mu}{r^2}\partial_\mu[Q_l{}^\beta,[Q_{k\alpha},\Phi^I(x)]_\pm]_\mp+ \\
&+i\epsilon^{ijkl}(S^{\mu\nu})^I{}_J\frac{r_\mu}{r^2}\partial_\nu[Q_l{}^\beta,[Q_{k\alpha},\Phi^J(x)]_\pm]_\mp+ \\
&+i\epsilon^{ijkl}\epsilon_{\mu\nu\rho\lambda}(S^{\mu\nu})^I{}_J\frac{r^\rho}{r^2}\partial^\lambda[Q_l{}^\beta,[Q_{k\alpha},\Phi^J(x)]_\pm]_\mp+ \\
&+i\epsilon^{ijkl}(\sigma_{\mu\nu})_\alpha{}^\beta(S^{\mu\lambda})^I{}_J(\sigma^\nu{}_\lambda)_\gamma{}^\delta\frac{r^\rho}{r^2}\partial_\rho[Q_l{}^\gamma,[Q_{k\delta},\Phi^J(x)]_\pm]_\mp+ \\
&+i\epsilon^{ijkl}\epsilon^{\mu\nu\rho\sigma}(\sigma_{\mu\omega})_\alpha{}^\beta(S_{\nu\lambda})^I{}_J(\sigma_{\rho\tau})_\gamma{}^\delta\frac{r^\omega r^\tau r^\lambda}{r^4}\partial_\sigma[Q_l{}^\gamma,[Q_{k\delta},\Phi^J(x)]_\pm]_\mp+ \\
&+i\epsilon^{ijkl}(\sigma_{\mu\nu})_\alpha{}^\beta(S^{\mu\lambda})^I{}_J(\sigma_{\rho\tau})_\gamma{}^\delta\frac{r^\nu r^\tau r_\lambda}{r^4}\partial^\rho[Q_l{}^\gamma,[Q_{k\delta},\Phi^J(x)]_\pm]_\mp+ \\
&+\text{permutations}\,, \qquad \text{(C.6)}
\end{aligned}$$

where the permutations include, for example,

$$i\epsilon^{ijkl}(\sigma_{\mu\nu})_\alpha{}^\beta(S^{\mu\rho})^I{}_J(\sigma_{\rho\lambda})_\gamma{}^\delta\frac{r^\nu}{r^2}\partial^\lambda[Q_l{}^\gamma,[Q_{k\delta},\Phi^J(x)]_\pm]_\mp\,. \qquad \text{(C.7)}$$

In this case all the terms are B type.

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(iii) $\frac{1}{z}\,\bar{\delta}^3\delta\Phi$**:**

$$
\begin{aligned}
\Sigma_\alpha{}^{\beta ij}(z)\Phi^I(x) \sim\ & \delta_\alpha{}^\beta\frac{r^\mu}{r^2}(\sigma_\mu)_{\gamma\dot\gamma}[\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\alpha},[\bar Q^{k\dot\gamma},[Q_k{}^\gamma,\Phi^I(x)]_\pm]_\mp]_\pm]_\mp +\\
&+(\sigma_{\mu\nu})_\alpha{}^\beta\frac{r_\lambda}{r^2}(\sigma^\nu)_{\gamma\dot\gamma}(\bar\sigma^{\mu\lambda})^{\dot\alpha}{}_{\dot\beta}[\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\beta},[\bar Q^{k\dot\gamma},[Q_k{}^\gamma,\Phi^I(x)]_\pm]_\mp]_\pm]_\mp +\\
&+i\delta_\alpha{}^\beta(S^{\mu\nu})^I{}_J\frac{r_\mu}{r^2}(\sigma_\nu)_{\gamma\dot\gamma}[\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\alpha},[\bar Q^{k\dot\gamma},[Q_k{}^\gamma,\Phi^J(x)]_\pm]_\mp]_\pm]_\mp +\\
&+i\delta_\alpha{}^\beta\epsilon_{\mu\nu\rho\lambda}(S^{\mu\nu})^I{}_J\frac{r^\rho}{r^2}(\sigma^\lambda)_{\gamma\dot\gamma}[\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\alpha},[\bar Q^{k\dot\gamma},[Q_k{}^\gamma,\Phi^J(x)]_\pm]_\mp]_\pm]_\mp +\\
&+i\delta_\alpha{}^\beta(S^{\mu\nu})^I{}_J\frac{r^\lambda}{r^2}(\sigma_{\mu\nu}\sigma_\lambda)_{\gamma\dot\gamma}[\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\alpha},[\bar Q^{k\dot\gamma},[Q_k{}^\gamma,\Phi^J(x)]_\pm]_\mp]_\pm]_\mp +\\
&+i(\sigma_{\mu\nu})_\alpha{}^\beta(S^{\mu\lambda})^I{}_J(\bar\sigma_{\rho\lambda})^{\dot\alpha}{}_{\dot\beta}\frac{r^\nu}{r^2}(\sigma^\rho)_{\gamma\dot\gamma}[\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\beta},[\bar Q^{k\dot\gamma},[Q_k{}^\gamma,\Phi^J(x)]_\pm]_\mp]_\pm]_\mp +\\
&+i\epsilon^{\mu\nu\rho\sigma}(\sigma_{\mu\omega})_\alpha{}^\beta(S_{\nu\tau})^I{}_J(\bar\sigma_{\rho\lambda})^{\dot\alpha}{}_{\dot\beta}\frac{r^\tau r^\omega r_\sigma}{r^4}\times\\
&\times(\sigma^\lambda)_{\gamma\dot\gamma}[\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\beta},[\bar Q^{k\dot\gamma},[Q_k{}^\gamma,\Phi^J(x)]_\pm]_\mp]_\pm]_\mp +\\
&+i(\sigma_{\mu\rho})_\alpha{}^\beta(S^{\mu\tau})^I{}_J(\bar\sigma_{\tau\nu})^{\dot\alpha}{}_{\dot\beta}\frac{r^\rho r^\lambda r^\nu}{r^4}\times\\
&\times(\sigma_\lambda)_{\gamma\dot\gamma}[\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\beta},[\bar Q^{k\dot\gamma},[Q_k{}^\gamma,\Phi^J(x)]_\pm]_\mp]_\pm]_\mp +\\
&+\text{permutations}\,.
\end{aligned}
\tag{C.8}
$$

In this case, there are only permutations of B type terms.

(iv) $\frac{1}{z}\,\delta^3\bar{\delta}\Phi$**:**

$$
\begin{aligned}
\Sigma_\alpha{}^{\beta ij}(z)\Phi^I(x) \sim\ & \epsilon^{ijkl}\frac{r^\mu}{r^2}(\sigma_\mu)_{\gamma\dot\gamma}[Q_l{}^\beta,[Q_{k\alpha},[Q_m{}^\gamma,[\bar Q^{m\dot\gamma},\Phi^I(x)]_\pm]_\mp]_\pm]_\mp +\\
&+\epsilon^{klm[i}\frac{r^\mu}{r^2}(\sigma_\mu)_{\gamma\dot\gamma}[Q_l{}^\beta,[Q_{k\alpha},[Q_m{}^\gamma,[\bar Q^{j]\dot\gamma},\Phi^I(x)]_\pm]_\mp]_\pm]_\mp +\\
&+i\epsilon^{ijkl}\frac{r_\mu}{r^2}(\sigma_\nu)_{\gamma\dot\gamma}(S^{\mu\nu})^I{}_J[Q_l{}^\beta,[Q_{k\alpha},[Q_m{}^\gamma,[\bar Q^{m\dot\gamma},\Phi^J(x)]_\pm]_\mp]_\pm]_\mp +\\
&+i\epsilon^{ijkl}\epsilon_{\mu\nu\rho\lambda}\frac{r^\rho}{r^2}(\sigma^\lambda)_{\gamma\dot\gamma}(S^{\mu\nu})^I{}_J[Q_l{}^\beta,[Q_{k\alpha},[Q_m{}^\gamma,[\bar Q^{m\dot\gamma},\Phi^J(x)]_\pm]_\mp]_\pm]_\mp +\\
&+i\epsilon^{ijkl}\frac{r^\mu}{r^2}(\sigma_{\rho\lambda}\sigma_\mu)_{\gamma\dot\gamma}(S^{\rho\lambda})^I{}_J[Q_l{}^\beta,[Q_{k\alpha},[Q_m{}^\gamma,[\bar Q^{m\dot\gamma},\Phi^J(x)]_\pm]_\mp]_\pm]_\mp +\\
&+i\epsilon^{ijkl}(\sigma_{\mu\nu})_\alpha{}^\beta(S^{\mu\lambda})^I{}_J(\sigma^\nu{}_\lambda)_\gamma{}^\delta\frac{r_\rho}{r^2}(\sigma^\rho)_{\theta\dot\theta}[Q_l{}^\gamma,[Q_{k\delta},[Q_m{}^\theta,[\bar Q^{m\dot\theta},\Phi^J(x)]_\pm]_\mp]_\pm]_\mp +\\
&+i\epsilon^{ijkl}\epsilon^{\mu\nu\rho\sigma}(\sigma_{\mu\lambda})_\alpha{}^\beta(S_{\nu\tau})^I{}_J(\sigma_{\rho\omega})_\gamma{}^\delta(\sigma_\sigma)_{\theta\dot\theta}\frac{r^\omega r^\lambda r^\tau}{r^4}\times\\
&\times[Q_l{}^\gamma,[Q_{k\delta},[Q_m{}^\theta,[\bar Q^{m\dot\theta},\Phi^J(x)]_\pm]_\mp]_\pm]_\mp +\\
&+i\epsilon^{ijkl}(\sigma_{\mu\lambda})_\alpha{}^\beta(S^{\mu\tau})^I{}_J(\sigma_{\nu\tau})_\gamma{}^\delta(\sigma_\omega)_{\theta\dot\theta}\frac{r^\omega r^\lambda r^\nu}{r^4}\times\\
&\times[Q_l{}^\gamma,[Q_{k\delta},[Q_m{}^\theta,[\bar Q^{m\dot\theta},\Phi^J(x)]_\pm]_\mp]_\pm]_\mp +\\
&+i\delta_\alpha{}^\beta\epsilon^{klm(i}(\sigma_{\mu\nu})_\gamma{}^\delta(S^{\mu\nu})^I{}_J\frac{r_\rho}{r^2}(\sigma^\rho)_{\theta\dot\theta}[Q_l{}^\gamma,[Q_{k\delta},[Q_m{}^\theta,[\bar Q^{j)\dot\theta},\Phi^J(x)]_\pm]_\mp]_\pm]_\mp +\\
&+i\delta_\alpha{}^\beta\epsilon^{\mu\rho\tau\sigma}\epsilon^{klm(i}(S_{\mu\nu})^I{}_J(\sigma_{\rho\lambda})_\gamma{}^\delta(\sigma_\tau)_{\theta\dot\theta}\frac{r^\nu r^\lambda r_\sigma}{r^4}\times\\
&\times[Q_l{}^\gamma,[Q_{k\delta},[Q_m{}^\theta,[\bar Q^{j)\dot\theta},\Phi^J(x)]_\pm]_\mp]_\pm]_\mp +\\
&+i\delta_\alpha{}^\beta\epsilon^{klm(i}(S_{\mu\nu})^I{}_J(\sigma^{\rho\nu})_\gamma{}^\delta(\sigma^\lambda)_{\theta\dot\theta}\frac{r^\mu r_\lambda r_\rho}{r^4}[Q_l{}^\gamma,[Q_{k\delta},[Q_m{}^\theta,[\bar Q^{j)\dot\theta},\Phi^J(x)]_\pm]_\mp]_\pm]_\mp +
\end{aligned}
$$

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$$+\text{permutations}\,, \tag{C.9}$$

Note that every $\mathcal{E}$ term is an $S_{\mu\nu}$ term. Also this is the only case where the permutations include both $\mathcal{E}$ and B type terms. For example, the $\mathcal{E}$ type term

$$i\delta_\alpha{}^\beta \epsilon^{klm(i}(\sigma_{\mu\nu})_\gamma{}^\delta (S^{\mu\lambda})^I{}_J \frac{r^\nu}{r^2}(\sigma_\lambda)_{\theta\dot\theta}[Q_l{}^\gamma,[Q_{k\delta},[Q_m{}^\theta,[\bar{Q}^{j)\dot\theta},\Phi^J(x)]_\pm]_\mp]_\pm]_\mp\,. \tag{C.10}$$

So including all these additional terms, we have the complete OPE of $\delta^2\mathcal{O}_2$. The leading terms are given in (3.39), while the subleading terms are given in (C.3), (C.6), (C.8) and (C.9).

D. Some consistency checks of the $\delta^2\mathcal{O}_2$ OPE

Since the $\delta^2\mathcal{O}_2$ OPE plays an important role in our analysis, we will perform some consistency checks of this OPE here. The checks will involve computing the OPE of $\mathcal{E}$ with selected operators.

D.1 $\mathcal{E}(z)\bar\Lambda(x)$ in free field theory

We consider the $\bar\Lambda = \bar\delta^3\mathcal{O}_2$ operator which is given by

$$\bar\Lambda_i{}^{\dot\alpha} = -\frac{1}{g^2_{\mathrm{YM}}}(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}F_{\mu\nu}\bar\lambda_i{}^{\dot\beta}\,, \tag{D.1}$$

and we want to compute

$$\mathcal{E}^{kl}(z)\bar\Lambda_i{}^{\dot\alpha}(x)\,, \tag{D.2}$$

in the free field theory, where $\mathcal{E}$ is given by

$$\mathcal{E}^{ij} = \frac{1}{g^2_{\mathrm{YM}}}\lambda^i\lambda^j\,. \tag{D.3}$$

A straightforward calculation yields that

$$\mathcal{E}^{kl}(z)\bar\Lambda_i{}^{\dot\alpha}(x) \sim -\frac{i}{8\pi^2 g^2_{\mathrm{YM}}}\delta_i{}^l(\bar\sigma^{\mu\nu}\epsilon)^{\dot\alpha\dot\beta}\partial_{\gamma\dot\beta}\frac{1}{(z-x)^2}\left(F_{\mu\nu}\lambda^{k\gamma}\right)(x) - (k\leftrightarrow l)\,, \tag{D.4}$$

to leading order, where we have used the free fermion propagator (2.20). We now show that the leading term in (3.39) given by

$$\mathcal{E}^{kl}(z)\bar\Lambda_i{}^{\dot\alpha}(x) \sim \frac{1}{(z-x)^2}[\bar{Q}^k{}_{\dot\beta},\{\bar{Q}^{l\dot\beta},\bar\Lambda_i{}^{\dot\alpha}(x)\}]\,, \tag{D.5}$$

gives us (D.4) on integrating by parts. Since we will eventually integrate over z, equivalence up to total derivatives is sufficient for us.

Using the relations

$$\{\bar{Q}^{i\dot\alpha},\bar\Lambda_j{}^{\dot\beta}\} = \delta_j{}^i\epsilon^{\dot\alpha\dot\beta}\mathcal{O}_{\bar\tau}, \qquad [\bar{Q}^i,\bar{\mathcal{O}}_\tau] = \partial^\mu\bar{J}^i_\mu + 2\bar\sigma^\mu\sigma^\nu\partial_\mu\bar{J}^i_\nu\,, \tag{D.6}$$

we see that (D.5) gives

$$\mathcal{E}^{kl}(z)\bar{\Lambda}_i{}^{\dot\alpha}(x) \sim -\frac{2\delta_i{}^l}{(z-x)^2}(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\partial_\mu \bar{J}_\nu^{k\dot\beta}(x) - (k \leftrightarrow l)\,. \tag{D.7}$$

Now $\bar{J}_\mu^{i\dot\alpha}$ is given by

$$\bar{J}_\mu^{i\dot\alpha} = -\frac{1}{g_{\rm YM}^2}\left(F_{\rho\sigma}(\bar\sigma^{\rho\sigma}\bar\sigma_\mu\lambda^i)^{\dot\alpha} + 2i\varphi^{ij}\overset{\leftrightarrow\mu}{\partial}\bar\lambda_j{}^{\dot\alpha} + \frac{4}{3}i\partial^\nu(\varphi^{ij}\bar\sigma_{\mu\nu}\bar\lambda_j)^{\dot\alpha}\right). \tag{D.8}$$

Again, a straightforward calculation leads to

$$\mathcal{E}^{kl}(z)\bar\Lambda_i{}^{\dot\alpha}(x) \sim \frac{1}{2g_{\rm YM}^2}\delta_i{}^l(\bar\sigma^{\mu\nu}\epsilon)^{\dot\alpha\dot\beta}\partial_{\gamma\dot\beta}\frac{1}{(z-x)^2}\left(F_{\mu\nu}\lambda^{k\gamma}\right)(x) + (k\leftrightarrow l)\,, \tag{D.9}$$

where we have integrated by parts. The total contribution from the second and the third terms in (D.8) cancels. So we see that up to an overall numerical factor, (D.4) and (D.9) match exactly — demonstrating the existence of the leading term in the $\mathcal{E}$ OPE in (3.39).

D.2 $\mathcal{E}(z)\bar\delta\mathcal{O}_2(x)$ in free field theory

We consider this example because it is a case where the subleading terms in the $\mathcal{E}$ OPE are needed for agreement with direct free field computations. Start with the $\bar\delta\mathcal{O}_2$ operator given by

$$\bar\chi^{ij}{}_k = \frac{1}{2g_{\rm YM}^2}\epsilon^{ijmn}\left(\bar\varphi_{mn}\bar\lambda_k + \bar\varphi_{kn}\bar\lambda_m\right), \tag{D.10}$$

to leading order and compute in free field theory

$$\mathcal{E}^{kl}(z)\bar\chi^{ij\dot\alpha}{}_m(x) \sim -\frac{i}{4\pi^2 g_{\rm YM}^2}\epsilon^{ijpq}\partial^{\dot\alpha\alpha}\frac{1}{(z-x)^2}\left(\delta_m{}^{(k}\lambda^{l)}\bar\varphi_{pq} + \delta_p{}^{(k}\lambda^{l)}\bar\varphi_{mq}\right)_\alpha(x)\,, \tag{D.11}$$

This expression can also be written as

$$\mathcal{E}^{kl}(z)\bar\chi^{ij\dot\alpha}{}_m(x) \sim -\frac{i}{4\pi^2 g_{\rm YM}^2}\partial^{\dot\alpha\alpha}\frac{1}{(z-x)^2}\left(3\delta_m{}^{(k}\lambda^{l)}\varphi^{ij} + 2\delta_m{}^{[i}\varphi^{j](k}\lambda^{l)}\right)_\alpha(x) \tag{D.12}$$

using $\bar\varphi_{ij} = \frac{1}{2}\epsilon_{ijkl}\varphi^{kl}$. Now the leading term in the OPE (3.39) is schematically $\frac{1}{z^2}\bar\delta^2\bar\chi \sim \frac{1}{z^2}\bar\Lambda$, which is not the correct term; hence we can conclude its coefficient vanishes. Similarly, all the coefficients of all terms in (C.3), (C.6) and (C.8) must vanish as well.

Finally, we consider the terms in (C.9), whose contribution $\frac{1}{z^2}\delta^3\bar\delta\bar\chi$ contains J and χ. So we need to find the terms in (C.9) that reproduce (D.12) — these terms must have non-vanishing coefficients. For this example, note that

$$(S_{\mu\nu})^I{}_J\Phi^J = (\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\bar\chi^{ij\dot\beta}{}_m\,. \tag{D.13}$$

Many of the $\mathcal{E}$ type $S_{\mu\nu}$ term contributions in (C.9) vanish identically for this choice of $S_{\mu\nu}$. For simplicity, we consider a particular non-vanishing term in (C.9). We shall see that this term is sufficient to reproduce the structures appearing in (D.12). However, we will not calculate any precise coefficients since other terms in (C.9) can also give rise to the terms of (D.12) (also, the algebra is quite cumbersome).

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Consider the term in (C.9) given by

$$\begin{aligned}\mathcal{E}^{kl}(z)\bar\chi^{ij\dot\alpha}{}_m(x) \sim{}& i\epsilon^{\mu\rho\tau\omega}\epsilon^{pqs(k}(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}(\sigma_{\rho\sigma})_\gamma{}^\delta(\sigma_\tau)_{\theta\dot\theta}\\ &\times\frac{r^\nu r^\sigma r_\omega}{r^4}[Q_q{}^\gamma,\{Q_{p\delta},[Q_s{}^\theta,\{\bar Q^{l)\dot\theta},\bar\chi^{\dot\beta ij}{}_m(x)\}]\}]\,. \end{aligned}\tag{D.14}$$

In order to proceed, we need the supersymmetry transformations from (2.5)

$$\begin{aligned}\{Q_l{}^\alpha,\bar\chi^{ij\dot\alpha}{}_k\} ={}& -\frac{3}{4}i(\bar\sigma_\mu)^{\dot\alpha\alpha}R^{\mu[i}{}_k\delta^{j]}{}_l+\frac{3}{2}i(\bar\sigma^\mu)^{\dot\alpha\alpha}\partial_\mu Q^{ij}{}_{kl}\\ &-\frac{3}{16}i(\bar\sigma_\mu)^{\dot\alpha\alpha}\delta^{[i}{}_kR^{\mu j]}{}_l+\frac{3}{4}i(\bar\sigma^\mu)^{\dot\alpha\alpha}\delta^{[i}{}_k\partial_\mu Q^{j]m}{}_{ml}\,,\end{aligned}\tag{D.15}$$

$$[Q_i,T_{\mu\nu}] = \sigma_{(\mu|\rho|}\partial^\rho J_{\nu)i}\,,\tag{D.16}$$

$$[Q_{l\alpha},R^{\mu i}{}_j] = -\delta_l{}^iJ^\mu_{j\alpha}+\frac{1}{4}\delta_j{}^iJ^\mu_{l\alpha}+\frac{8}{3}i(\sigma^{\mu\nu}\partial_\nu\chi^i{}_{jl})_\alpha\,,\tag{D.17}$$

as well as (3.3), (3.4), (3.5), (3.6) and (3.7). We then calculate (D.14) with a liberal use of integration by parts. After some tedious calculations, we see that there are only three distinct structures that contribute:

$$\epsilon^{ijp(k}\chi^{l)}{}_{mp\alpha}(x)\partial^{\dot\alpha\alpha}\frac{1}{(z-x)^2}\,,\tag{D.18}$$

$$\frac{1}{3}\delta_m{}^{[i}\epsilon^{j]pq(k}\chi^{l)}{}_{pq\alpha}(x)\partial^{\dot\alpha\alpha}\frac{1}{(z-x)^2}\,,\tag{D.19}$$

and

$$\epsilon^{ijp(k}\delta_m{}^{l)}\partial_\nu^{-1}J^\nu_{p\alpha}(x)\partial^{\dot\alpha\alpha}\frac{1}{(z-x)^2}\,.\tag{D.20}$$

We will explain the meaning of the formal expression (D.20) momentarily. It is useful to rewrite the χ term contributions using the relations

$$\epsilon^{ijp(k}\chi^{l)}{}_{mp} = -\frac{1}{g^2_{\rm YM}}\left(\varphi^{ij}\lambda^{(k}\delta_m{}^{l)}+3\delta_m{}^{[i}\varphi^{j](k}\lambda^{l)}-\lambda^{[i}\varphi^{j](k}\delta_m{}^{l)}\right),\tag{D.21}$$

and

$$\frac{1}{3}\delta_m{}^{[i}\epsilon^{j]pq(k}\chi^{l)}{}_{pq} = \frac{1}{g^2_{\rm YM}}\left(\delta_m{}^{[i}\varphi^{j](k}\lambda^{l)}\right).\tag{D.22}$$

(The possibility

$$\epsilon^{ijpq}\chi^{(k}{}_{pq}\delta_m{}^{l)} = \frac{1}{g^2_{\rm YM}}\left(\varphi^{ij}\lambda^{(k}\delta_m{}^{l)}-\lambda^{[i}\varphi^{j](k}\delta_m{}^{l)}\right)\tag{D.23}$$

is not linearly independent.)

We now consider the formal expression (D.20) and evaluate it. Using the definition of J in (2.10), we see that

$$\partial_\nu^{-1}J^\nu_p = \frac{1}{g^2_{\rm YM}}\left(\frac{i}{2}\bar\varphi_{pq}\lambda^q-4i\partial_\nu^{-1}(\bar\varphi_{pq}\partial^\nu\lambda^q)-\partial_\nu^{-1}(F_{\rho\sigma}\sigma^{\rho\sigma}\sigma^\nu\bar\lambda_p)\right),\tag{D.24}$$

using $\partial_\mu\partial_\nu^{-1}=\eta_{\mu\nu}$. Now the second term in this expression, when inserted in (D.20), vanishes on integrating by parts because on-shell

$$\partial^{\dot\alpha\alpha}\frac{1}{(z-x)^2}\partial_\nu^{-1}\left(\bar\varphi_{pq}\partial^\nu\lambda^q\right)(x) = \frac{1}{(z-x)^2}\left(\bar\varphi_{pq}\partial^{\dot\alpha\alpha}\lambda^q{}_\alpha\right)(x)=0\,.\tag{D.25}$$

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Similarly, the third term when inserted in (D.20) also vanishes on integrating by parts because

$$\partial^{\dot\alpha\alpha}\frac{1}{(z-x)^2}\partial_\nu^{-1}(F_{\rho\sigma}(\sigma^{\rho\sigma}\sigma^\nu\bar\lambda_p)_\alpha) = \frac{1}{(z-x)^2}(\bar\sigma_\mu)^{\dot\alpha\alpha}(F_{\rho\sigma}(\sigma^{\rho\sigma}\sigma^\mu\bar\lambda_p)_\alpha) = (\sigma^{\rho\sigma})_\alpha{}^\alpha\times\cdots$$
$$= 0. \tag{D.26}$$

So we see that (D.20) is a well-defined local operator given by

$$\epsilon^{ijp(k}\delta_m{}^{l)}\partial_\nu^{-1}J^\nu_{p\alpha}(x)\partial^{\dot\alpha\alpha}\frac{1}{(z-x)^2} = \frac{i}{2g^2_{\rm YM}}\epsilon^{ijp(k}\delta_m{}^{l)}\left(\bar\varphi_{pq}\lambda^q{}_\alpha\right)(x)\partial^{\dot\alpha\alpha}\frac{1}{(z-x)^2}. \tag{D.27}$$

It is more useful to simplify this using the relation

$$\epsilon^{ijp(k}\delta_m{}^{l)}\left(\bar\varphi_{pq}\lambda^q{}_\alpha\right) = \left(\varphi^{ij}\delta_m{}^{(k}\lambda^{l)} + 2\lambda^{[i}\varphi^{j](k}\delta_m{}^{l)}\right). \tag{D.28}$$

From the expressions (D.21), (D.22) and (D.28), it is easy to see that we get all the terms in (D.12). This provides constraints on the OPE coefficients from the terms (C.9). In fact, we can write (D.12) as

$$\mathcal{E}^{kl}(z)\bar\chi^{ij\dot\alpha}{}_m(x) \sim \frac{3i}{8\pi^2}\partial^{\dot\alpha\alpha}\frac{1}{(z-x)^2}\Bigg(\epsilon^{ijp(k}\chi^{l)}{}_{mp} + \frac{20}{9}\delta_m{}^{[i}\epsilon^{j]pq(k}\chi^{l)}{}_{pq} +$$
$$+ i\epsilon^{ijp(k}\delta_m{}^{l)}\partial_\nu^{-1}J^\nu_p\Bigg)_\alpha(x). \tag{D.29}$$

We can now physically see the role of the last term in (D.29). Without this term, the right hand side of (D.29) would contain a term proportional to

$$\lambda^{[i}{}_\alpha\varphi^{j](k}\delta_m{}^{l)}\partial^{\dot\alpha\alpha}\frac{1}{(z-x)^2}, \tag{D.30}$$

which does not appear on the left hand side of (D.29). Also, $\mathcal{E}$ and $\bar\chi$ are both elements of the current multiplet which closes on-shell and so their OPE should involve only operators in the current multiplet, which is seen to be the case.

D.3 The $\mathcal{E}(z)T(x)$ OPE

The prior two checks of the $\delta^2\mathcal{O}_2$ OPE involved free field theory. This final check involves the full interacting theory. Let us consider the leading terms in the OPE of the stress tensor with $\mathcal{E}$, i.e., the $T(z)\mathcal{E}(x)$ OPE. Clearly, these terms (with z and x interchanged) should appear in the $\mathcal{E}(z)T(x)$ OPE and we will demonstrate that this is the case.

First consider the $T\mathcal{E}$ OPE. From (3.2) we see that, to leading order,

$$T_{\mu\nu}(z)\mathcal{E}^{ij}(x) \sim \mathcal{E}^{ij}(x)\partial_\mu\partial_\nu\frac{1}{(z-x)^2} + \mathcal{E}^{ij}(x)\eta_{\mu\nu}\delta^4(z-x). \tag{D.31}$$

Now consider the terms in the $\mathcal{E}T$ OPE. Since

$$\bar\delta^2 T\sim\bar B, \qquad \bar\delta^3\delta T\sim\bar B+\bar{\mathcal{E}}, \qquad \delta^3\bar\delta T\sim B+\mathcal{E},$$

we see that only terms from (C.9) can be non-zero again. All the coefficients from (3.39), (C.3), (C.6) and (C.8) must vanish. Among the terms in (C.9), we consider the contribution of two terms to the $\mathcal{E}T$ OPE. First consider

$$\mathcal{E}^{ij}(z)T_{\mu\nu}(x) \sim i\epsilon^{klm(i}(\sigma^{\lambda\rho})_\gamma{}^\delta \frac{r_\sigma}{r^2}(\sigma^\sigma)_{\theta\dot\theta} \times$$
$$\times\{Q_l{}^\gamma,[Q_{k\delta},\{Q_m{}^\theta,[\bar Q^{j)\dot\theta},(S_{\lambda\rho}T)_{\mu\nu}(x)]\}]\}\,, \tag{D.32}$$

where

$$(S_{\lambda\rho}T)_{\mu\nu} = -i(\eta_{\rho\mu}T_{\lambda\nu}-\eta_{\lambda\mu}T_{\rho\nu}+\eta_{\rho\nu}T_{\mu\lambda}-\eta_{\lambda\nu}T_{\mu\rho})\,. \tag{D.33}$$

Using the supersymmetry variations of (2.5),

$$[\bar Q^i,T_{\mu\nu}] = \bar\sigma_{\mu\lambda}\partial^\lambda \bar J^i_\nu + \bar\sigma_{\nu\lambda}\partial^\lambda \bar J^i_\mu\,, \tag{D.34}$$

$$\{Q_j{}^\alpha,\bar J^{i\dot\alpha}_\mu\} = (\bar\sigma^\nu)^{\dot\alpha\alpha}T_{\mu\nu}\delta_j{}^i + 2(\bar\sigma_\lambda\sigma_{\mu\nu}-\frac{1}{3}\bar\sigma_{\mu\nu}\bar\sigma_\lambda)^{\dot\alpha\alpha}\partial^\nu R^{\lambda i}{}_j\,, \tag{D.35}$$

$$[Q_k,R^{\mu i}{}_j] = -\delta_k{}^i J^\mu_j + \frac{1}{4}\delta_j{}^i J^\mu_k + \frac{8}{3}i\sigma^{\mu\nu}\partial_\nu\chi^i{}_{jk}\,, \tag{D.36}$$

and (3.4), we can evaluate (D.32). To simplify the computation, let us restrict to on-shell non-vanishing contributions. Any additional contributions will only change the value of the coefficients. On integrating by parts and using $\partial_\mu R^{\mu i}{}_j = 0$ (on-shell), we see that (D.32) becomes

$$\mathcal{E}^{ij}(z)T_{\mu\nu}(x) \sim \mathcal{E}^{ij}(x)\left(\partial_\mu\partial_\nu\frac{1}{(z-x)^2}+\pi^2\eta_{\mu\nu}\delta^4(z-x)\right). \tag{D.37}$$

Next consider another term in the $\mathcal{E}T$ OPE given by

$$\mathcal{E}^{ij}(z)T_{\mu\nu}(x) \sim \epsilon^{klm(i}(\sigma^{\sigma\rho})_\gamma{}^\delta(\sigma^\omega)_{\theta\dot\theta}\frac{r^\lambda r_\omega r_\sigma}{r^4} \times$$
$$\times\{Q_l{}^\gamma,[Q_{k\delta},\{Q_m{}^\theta,[\bar Q^{j)\dot\theta},(S_{\lambda\rho}T)_{\mu\nu}(x)]\}]\}\,. \tag{D.38}$$

Again evaluating this term gives

$$\mathcal{E}^{ij}(z)T_{\mu\nu}(x) \sim \mathcal{E}^{ij}(x)\left(3\partial_\mu\partial_\nu\frac{1}{(z-x)^2}-2\pi^2\eta_{\mu\nu}\delta^4(z-x)\right). \tag{D.39}$$

In this way, we recover the $T\mathcal{E}$ OPE structure from the $\mathcal{E}T$ OPE.

E. Some useful two-point functions

In this appendix, we list the results for the various two-point functions needed in section 4.2. All derivatives act on x. For cases with $\delta^2\bar\delta^2$ insertions:

$$\begin{aligned}
\langle\{Q_i{}^\alpha,[\bar Q^i{}_{\dot\alpha},\{\bar Q^{j\dot\alpha},[Q_{j\alpha},\bar{\mathcal{O}}(x)]\}]\}\mathcal{O}(y)\rangle &= -48\partial^2\langle\bar{\mathcal{O}}(x)\mathcal{O}(y)\rangle\,,\\
\langle\{\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\alpha},\{Q_i{}^\alpha,[Q_{j\alpha},\bar{\mathcal{O}}(x)]\}]\}\mathcal{O}(y)\rangle &= 32\partial^2\langle\bar{\mathcal{O}}(x)\mathcal{O}(y)\rangle\,,\\
(\sigma_{\mu\nu})_\alpha{}^\beta(\bar\sigma^{\mu\rho})^{\dot\alpha}{}_{\dot\beta}\langle\{Q_i{}^\alpha,[Q_{j\beta},\{\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\beta},\bar{\mathcal{O}}(x)]\}]\}\mathcal{O}(y)\rangle &= 8(\eta_{\rho\nu}\partial^2-4\partial_\nu\partial_\rho)\langle\bar{\mathcal{O}}(x)\mathcal{O}(y)\rangle\,,\\
(\sigma_{\mu\nu})_\alpha{}^\beta(\bar\sigma^{\mu\rho})^{\dot\alpha}{}_{\dot\beta}\langle\{\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\beta},\{Q_i{}^\alpha,[Q_{j\beta},\bar{\mathcal{O}}(x)]\}]\}\mathcal{O}(y)\rangle &= 8(\eta_{\rho\nu}\partial^2-4\partial_\nu\partial_\rho)\langle\bar{\mathcal{O}}(x)\mathcal{O}(y)\rangle\,,
\end{aligned}$$

$$(\sigma_{\mu\nu})_\alpha{}^\beta(\bar\sigma^{\mu\rho})^{\dot\alpha}{}_{\dot\beta}\langle\{Q_i{}^\alpha,[\bar Q^i{}_{\dot\alpha},\{\bar Q^{j\dot\beta},[Q_{j\beta},\bar{\mathcal O}(x)]\}]\}\mathcal O(y)\rangle \;=\; -4(\eta_{\rho\nu}\partial^2-4\partial_\nu\partial_\rho)\langle\bar{\mathcal O}(x)\mathcal O(y)\rangle.$$

For cases with $\delta^3\bar\delta^3$ insertions, we find that

$$\begin{aligned}
(\sigma_\mu)_{\beta\dot\beta}\langle\{Q_i{}^\alpha,[Q_{j\alpha},\{\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\alpha},\{\bar Q^{k\dot\beta},[Q_k{}^\beta,\bar{\mathcal O}(x)]\}]\}]\}\mathcal O(y)\rangle \;&=\; 128i\partial_\mu\partial^2\langle\bar{\mathcal O}(x)\mathcal O(y)\rangle\,,\\
(\sigma_\mu)_{\beta\dot\beta}\langle\{Q_i{}^\alpha,[\bar Q^i{}_{\dot\alpha},\{\bar Q^{j\dot\alpha},[\bar Q^{k\dot\beta},\{Q_k{}^\beta,[Q_{j\alpha},\bar{\mathcal O}(x)]\}]\}]\}\mathcal O(y)\rangle \;&=\; -256i\partial_\mu\partial^2\langle\bar{\mathcal O}(x)\mathcal O(y)\rangle\,,\\
(\sigma_\mu)_{\beta\dot\beta}\langle\{\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\alpha},\{\bar Q^{k\dot\beta},[Q_k{}^\beta,\{Q_i{}^\alpha,[Q_{j\alpha},\bar{\mathcal O}(x)]\}]\}]\}\mathcal O(y)\rangle \;&=\; 320i\partial_\mu\partial^2\langle\bar{\mathcal O}(x)\mathcal O(y)\rangle\,,
\end{aligned}\tag{E.1}$$

and,

$$\begin{aligned}
&(\sigma_{\mu\nu})_\alpha{}^\beta(\sigma^\nu)_{\gamma\dot\gamma}(\bar\sigma^{\mu\lambda})^{\dot\alpha}{}_{\dot\beta}\langle\{Q_i{}^\alpha,[Q_{j\beta},\{\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\beta},\{\bar Q^{k\dot\gamma},[Q_k{}^\gamma,\bar{\mathcal O}(x)]\}]\}]\}\mathcal O(y)\rangle =\\
&\qquad = -288i\partial_\lambda\partial^2\langle\bar{\mathcal O}(x)\mathcal O(y)\rangle\,,\\
&(\sigma_{\mu\nu})_\alpha{}^\beta(\sigma^\nu)_{\gamma\dot\gamma}(\bar\sigma^{\mu\lambda})^{\dot\alpha}{}_{\dot\beta}\langle\{Q_i{}^\alpha,[\bar Q^i{}_{\dot\alpha},\{\bar Q^{j\dot\beta},[\bar Q^{k\dot\gamma},\{Q_k{}^\gamma,[Q_{j\beta},,\bar{\mathcal O}(x)]\}]\}]\}\mathcal O(y)\rangle =\\
&\qquad = 384i\partial_\lambda\partial^2\langle\bar{\mathcal O}(x)\mathcal O(y)\rangle\,,\\
&(\sigma_{\mu\nu})_\alpha{}^\beta(\sigma^\nu)_{\gamma\dot\gamma}(\bar\sigma^{\mu\lambda})^{\dot\alpha}{}_{\dot\beta}\langle\{\bar Q^i{}_{\dot\alpha},[\bar Q^{j\dot\beta},\{\bar Q^{k\dot\gamma},[Q_k{}^\gamma,\{Q_i{}^\alpha,[Q_{j\beta},\bar{\mathcal O}(x)]\}]\}]\}\mathcal O(y)\rangle =\\
&\qquad = 48i\partial_\lambda\partial^2\langle\bar{\mathcal O}(x)\mathcal O(y)\rangle\,.
\end{aligned}\tag{E.2}$$

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